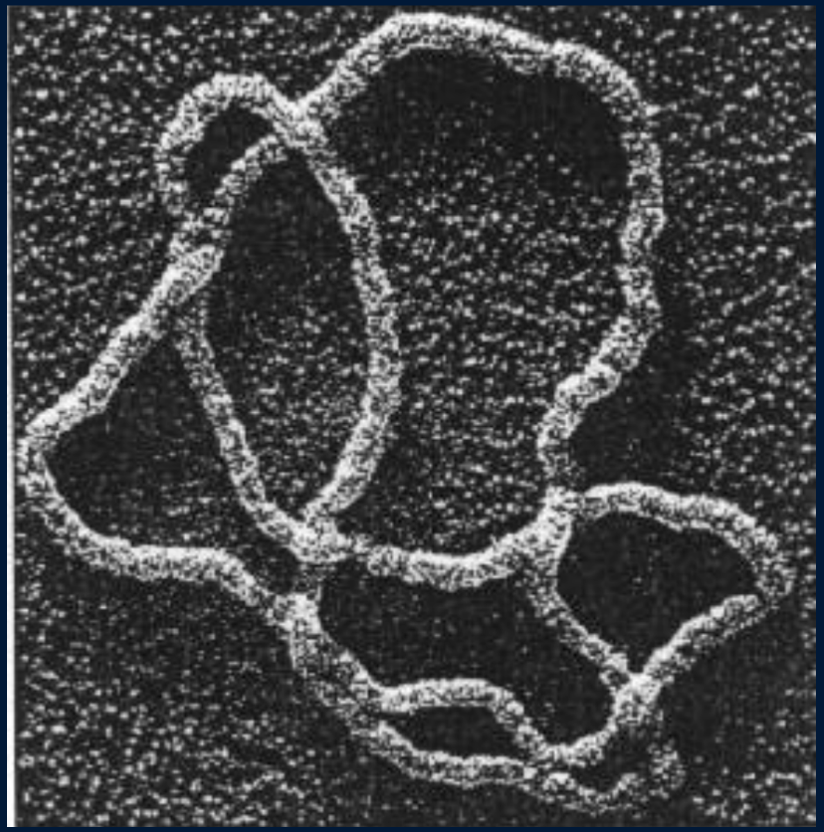
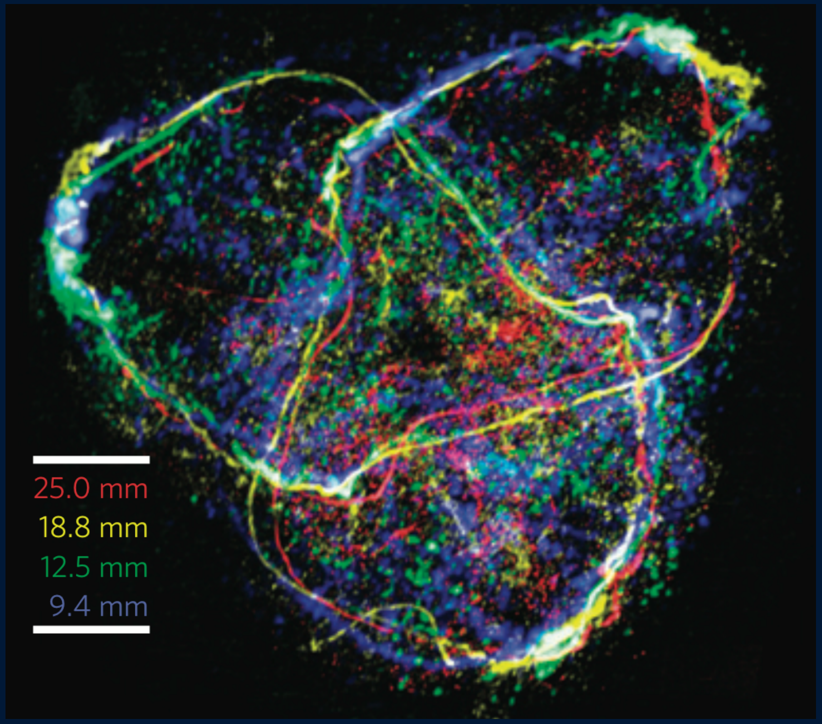
Knots, Fibrations and Physics

Knots!

Knotted DNA

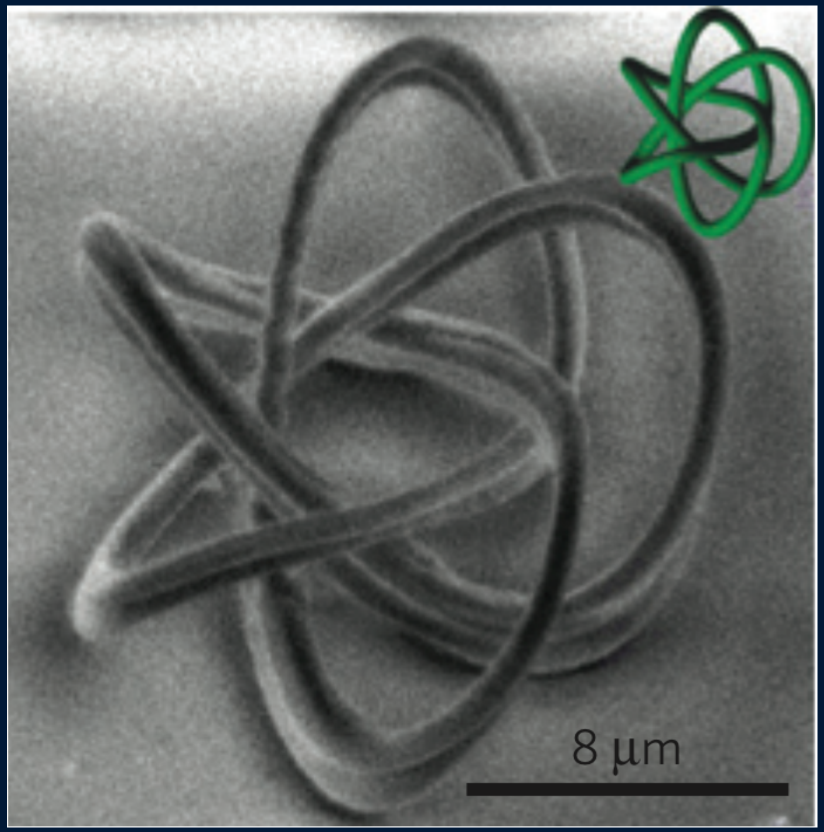


Knots!



Knotted Fluid Vortex

Knotted Liquid Crystal



Nematic Liquid Crystals

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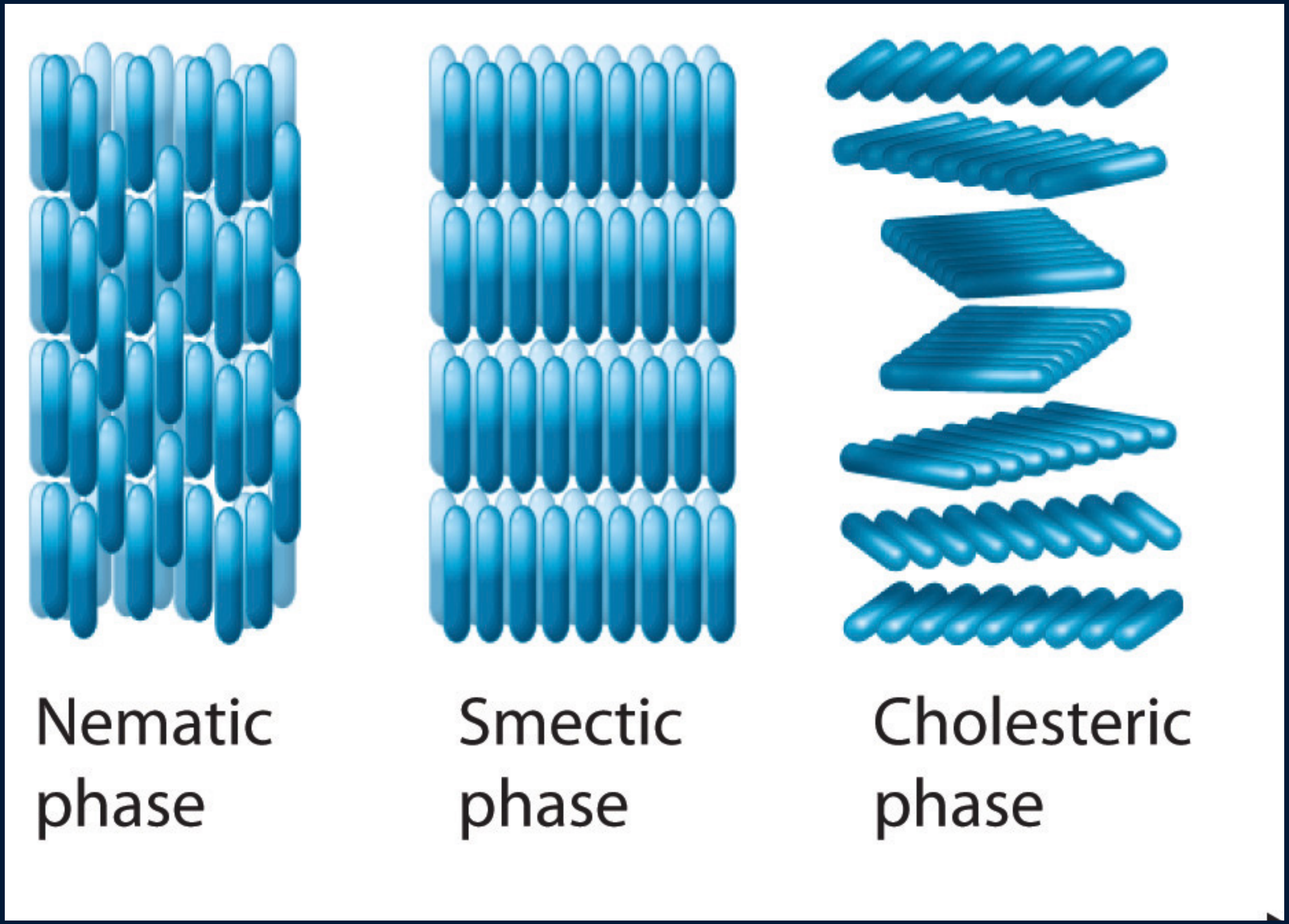
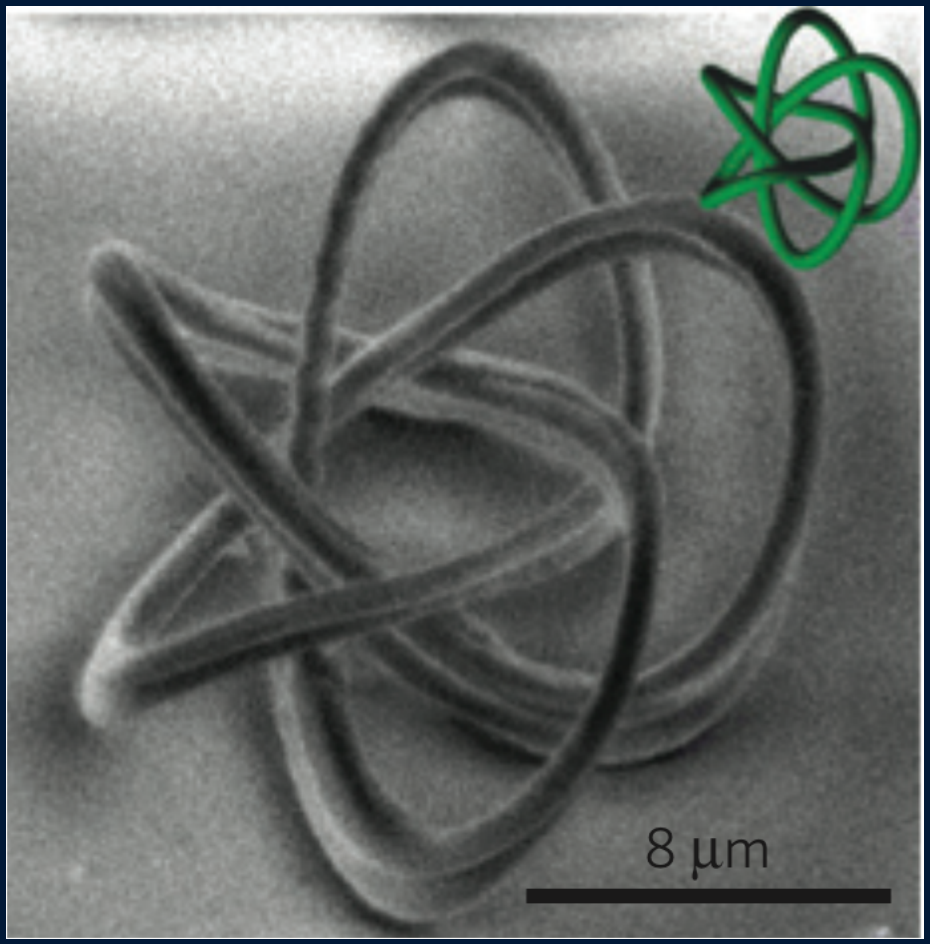


Knotted Nematics



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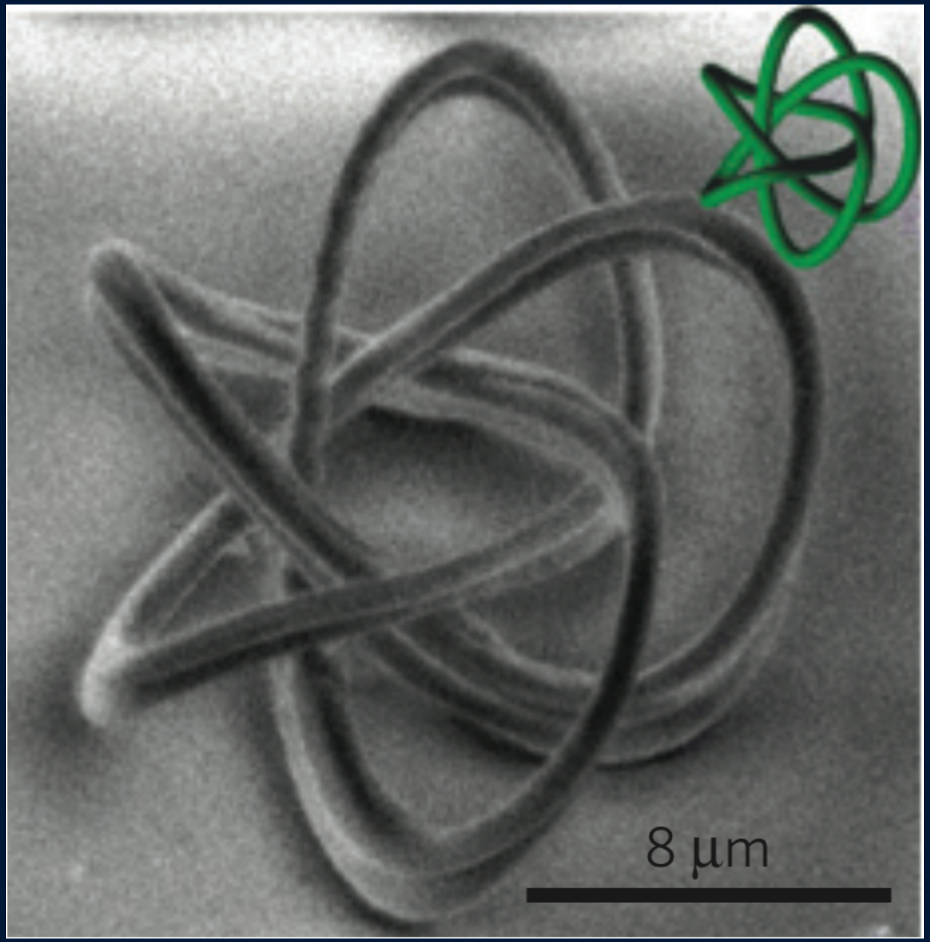

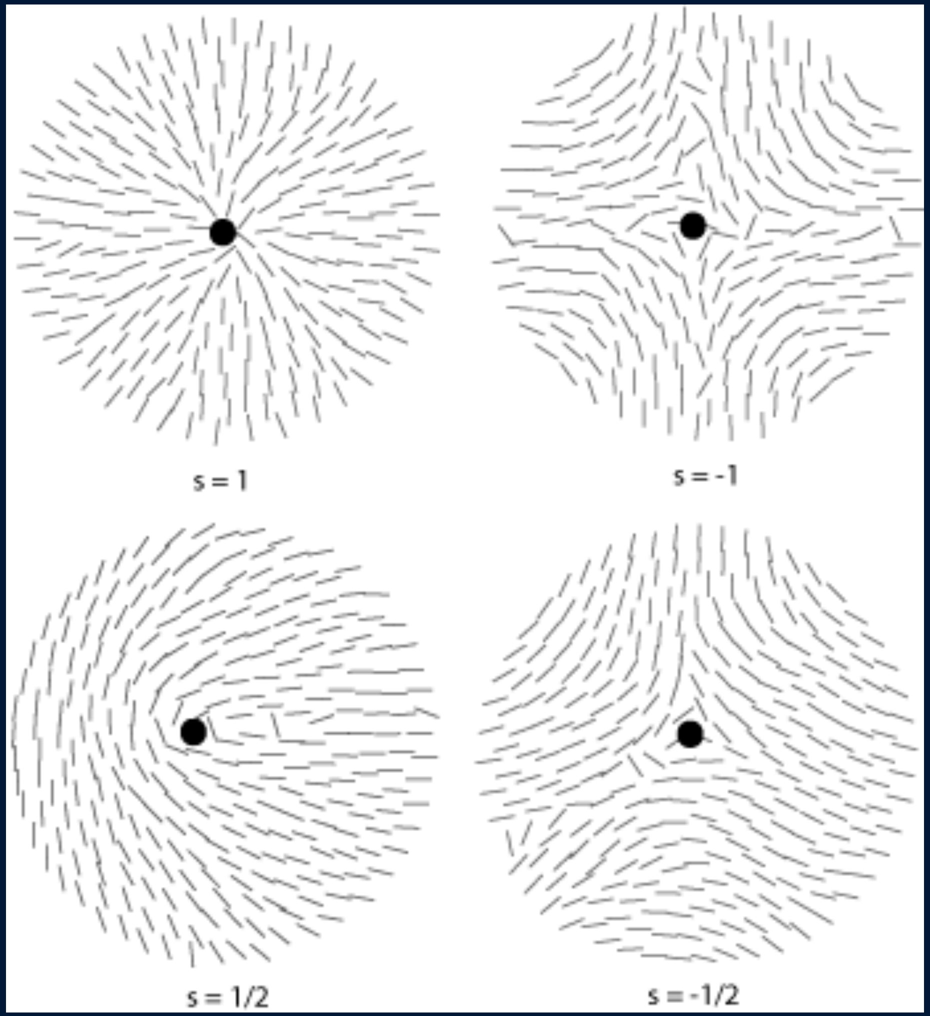

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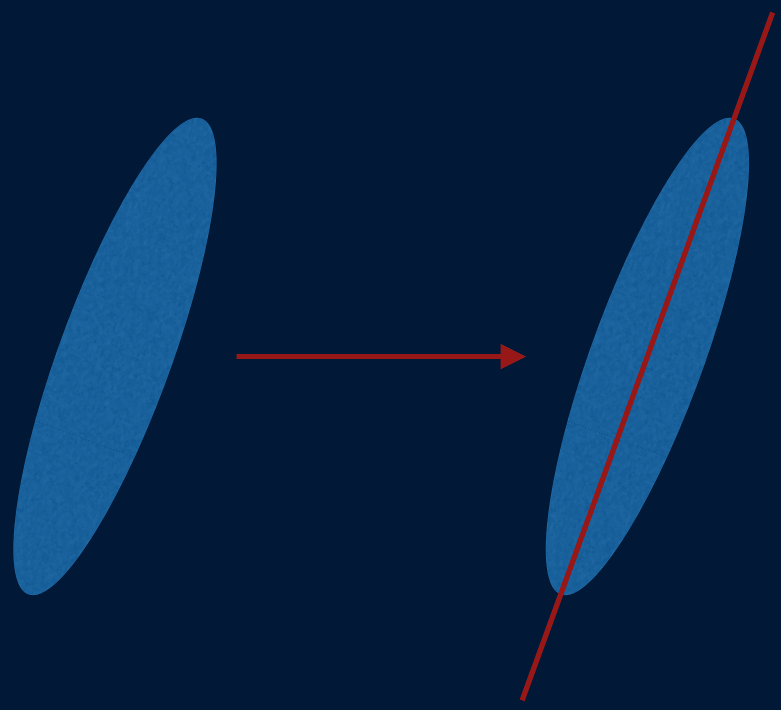

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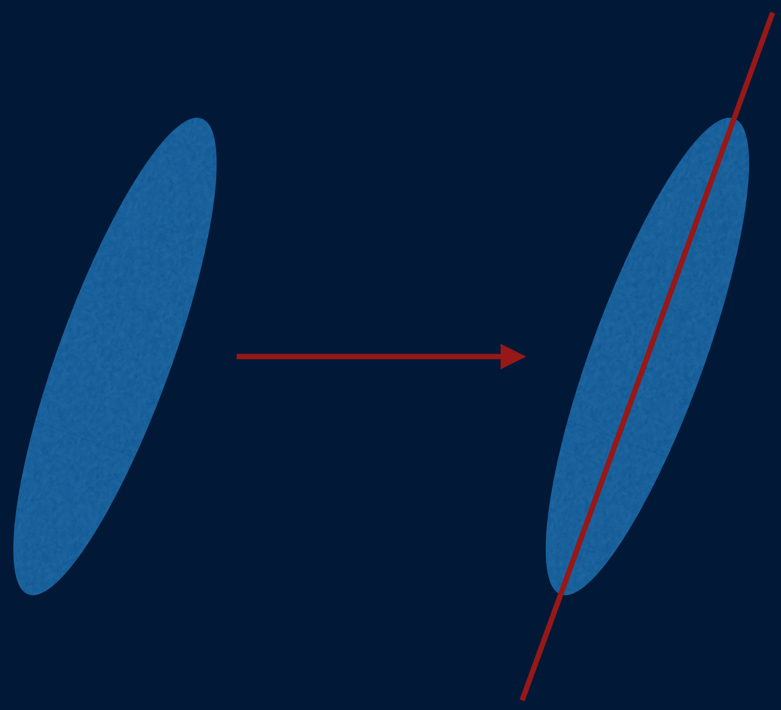

$U \subset \mathbb{R}^3$

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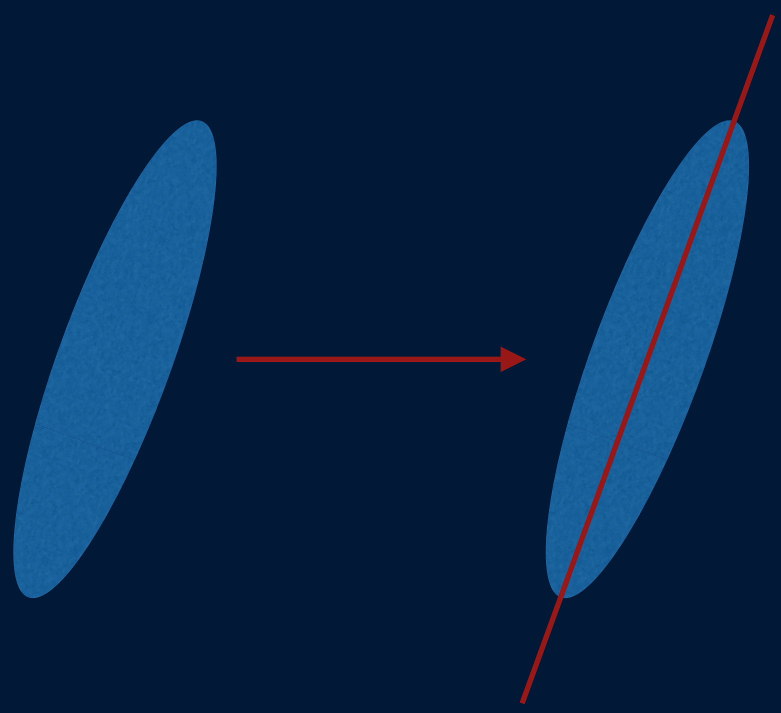

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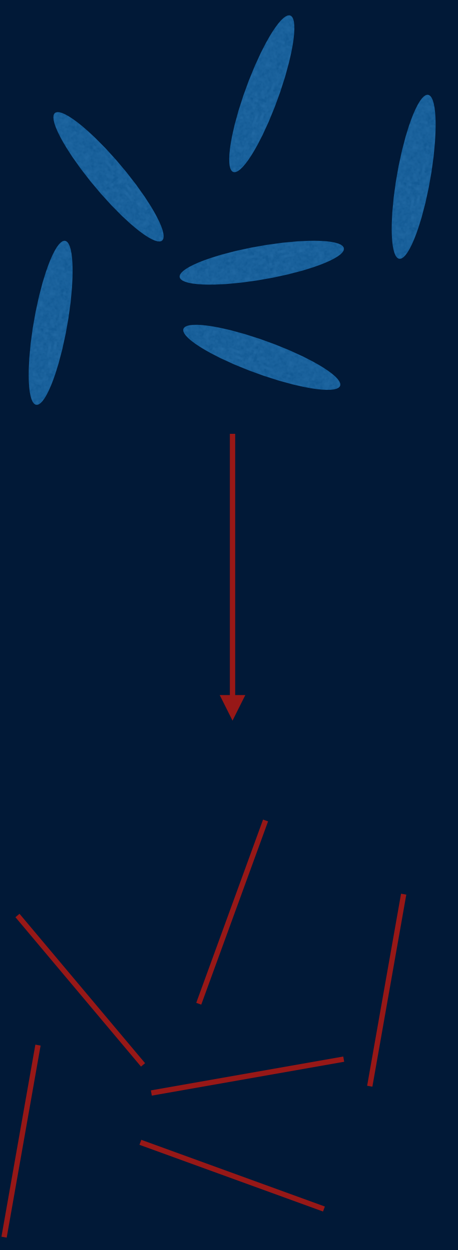

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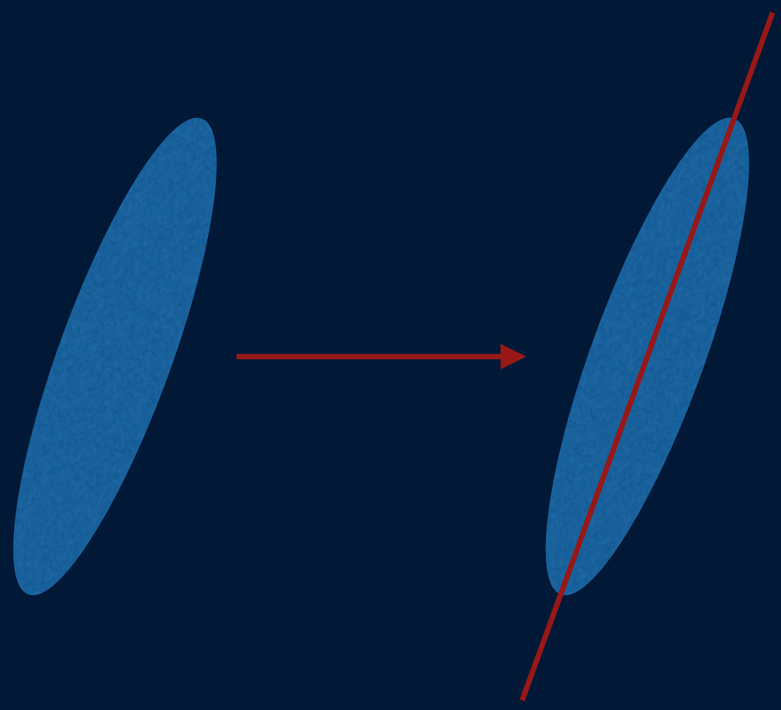

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$n : U \to \mathbb{RP}^2$

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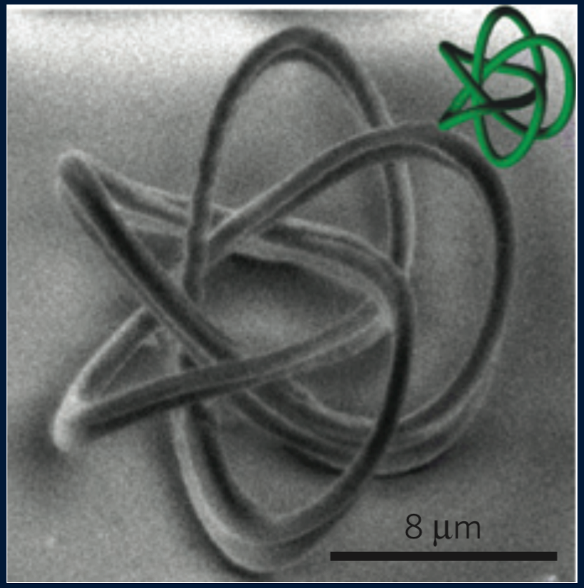

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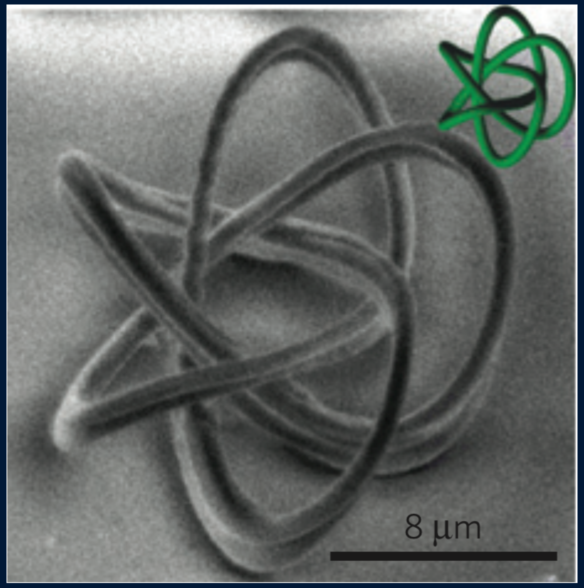

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So whats the plan?

Milnor Fibration Theorem

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If z^0 is any point of a complex hypersurface, $V = f^{-1}(0)$, where f is a polynomial, $f : \mathbb{C}^{n+1} \to \mathbb{C}$. If S_ϵ is a sufficiently small $(2n+1)$-dimensional sphere entered at z^0. Let $K = V \cap S_\epsilon$ then the mapping,

$$\phi : S_\epsilon \setminus K \to S^1 \qquad \phi z = \frac{f(z)}{\|f(z)\|}$$

is the projection of a smooth fibre bundle.
Each fibre, $F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\epsilon \setminus (V \cap S_\epsilon)$ is a smooth parallelisable 2n-dimensional manifold.

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$$S_\epsilon = \{(z_1, z_2) \in \mathbb{C}^2 \mid (z_1 - z_1^0)^2 + (z_2 - z_2^0)^2 = \epsilon\}$$

Lemma. *Let D_ϵ denote the closed disk consisting of all x with $\|x = x_0\| \leq \epsilon$. And let x_0 be a simple point or an isolated singular point of V. Then for small ϵ, the intersection of V with D_ϵ is homeomorphic to the cone over $K = V \cap S_\epsilon$.*

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So $K = V \cap S_\epsilon$ is a smooth manifold

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Each fibre, $F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\epsilon \backslash (V \cap S_\epsilon)$ is a smooth parallelisable 2-dimensional manifold.

Fibre Bundles

A *smooth fibre bundle* is a structure (E, B, π, F), where E, B, F are smooth manifolds and $\pi : E \to B$ is a smooth surjection such that:

For every $x \in E$, there is an open neighbourhood $U \subset B$ of $\pi(x)$ such that there is a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in a way that π agrees with the projection onto the first factor.

This last part can be summarised by saying the following diagram commutes.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
& U &
\end{tikzcd}$$

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$(S_\epsilon \setminus K, S^1, \phi, F)$, is a fibre bundle, with fibres $F_\theta = \phi^{-1}(e^{i\theta})$

Stereographic Projection

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Theorem. *If z_0 is an isolated critical point of f, then each fibre F_θ can be considered as the interior of a smooth manifold-with-boundary such that:*

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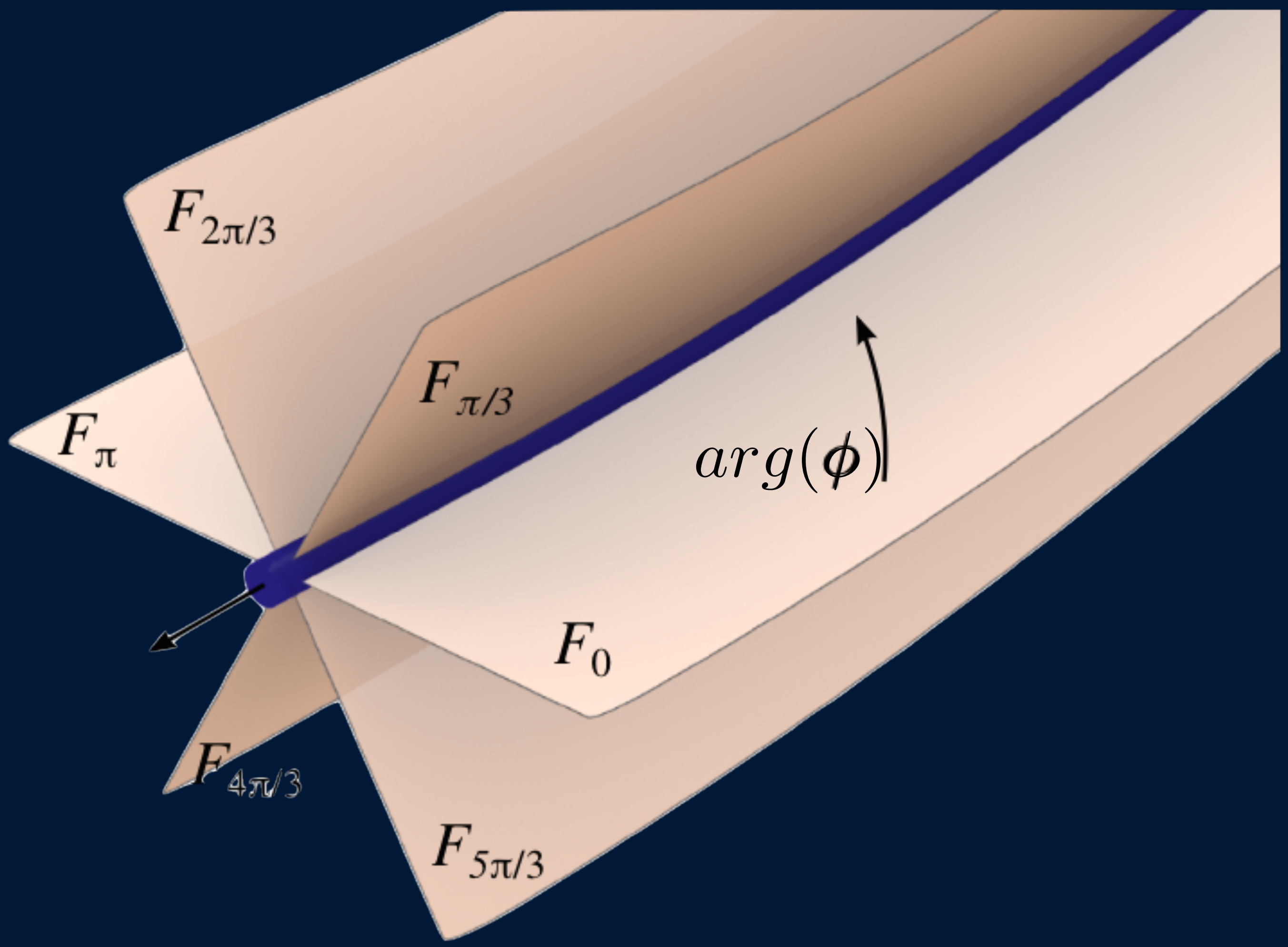

Knot Theory

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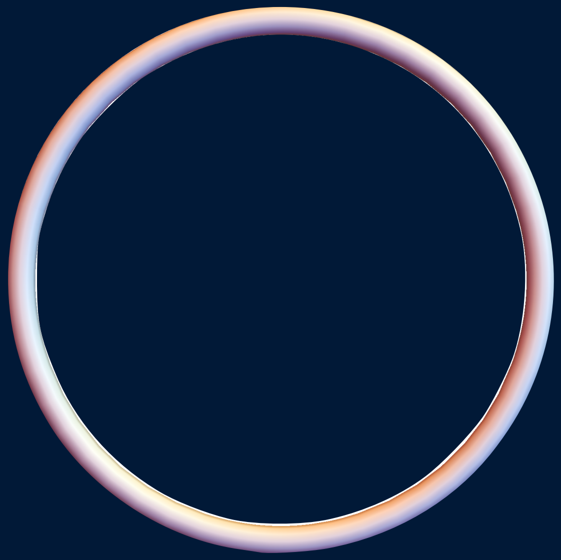
Unknot (0)



Knot Theory

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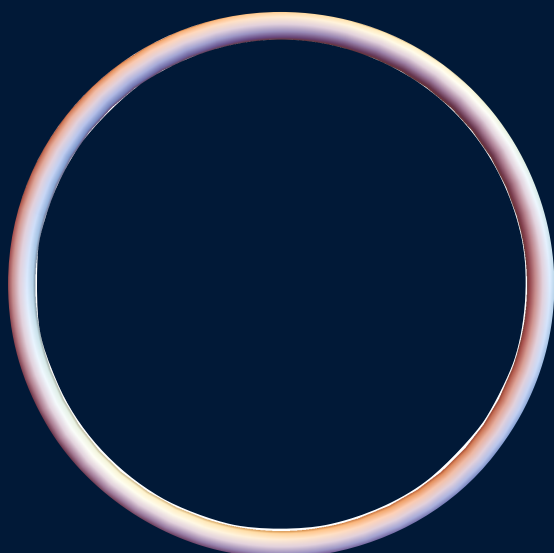


Trefoil (3_1)

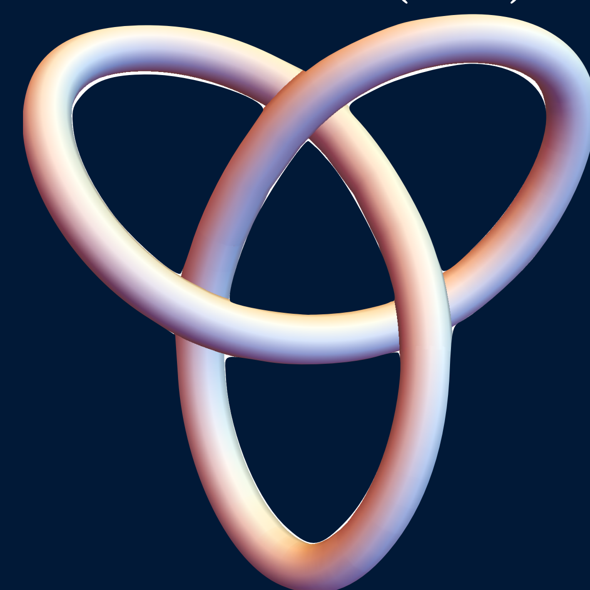
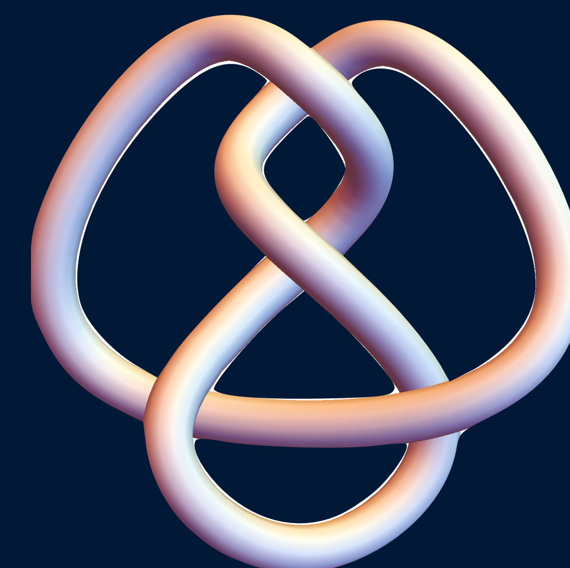
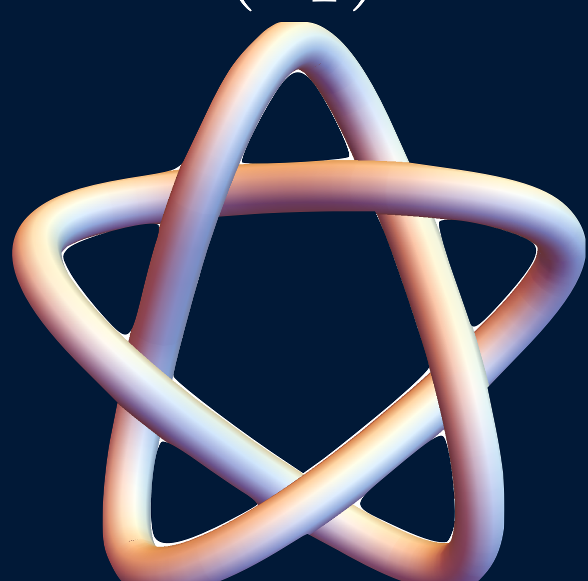


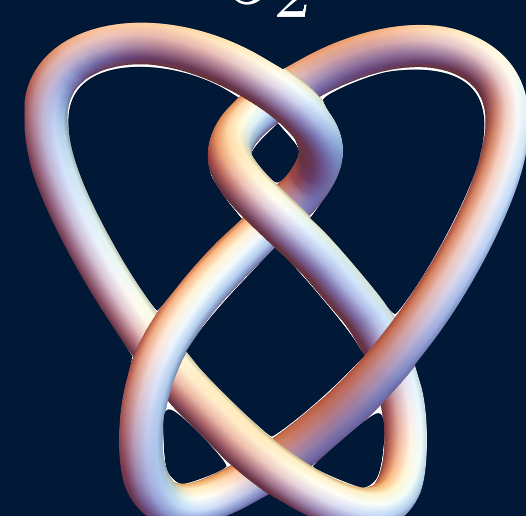
Figure 8 (4_1)



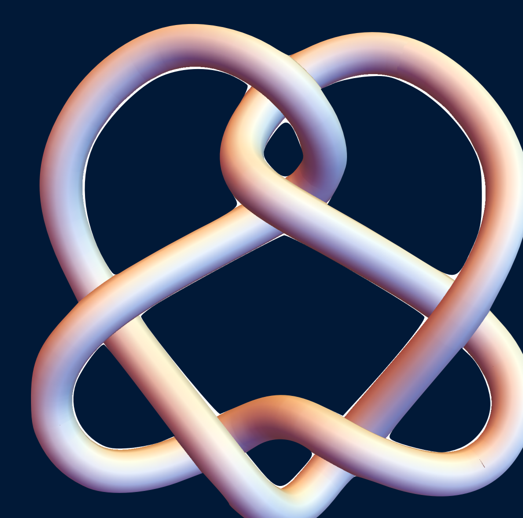
Solomon Seal (5_1)



5_2



6_1



Knot Theory

Algebraic Knot Theory

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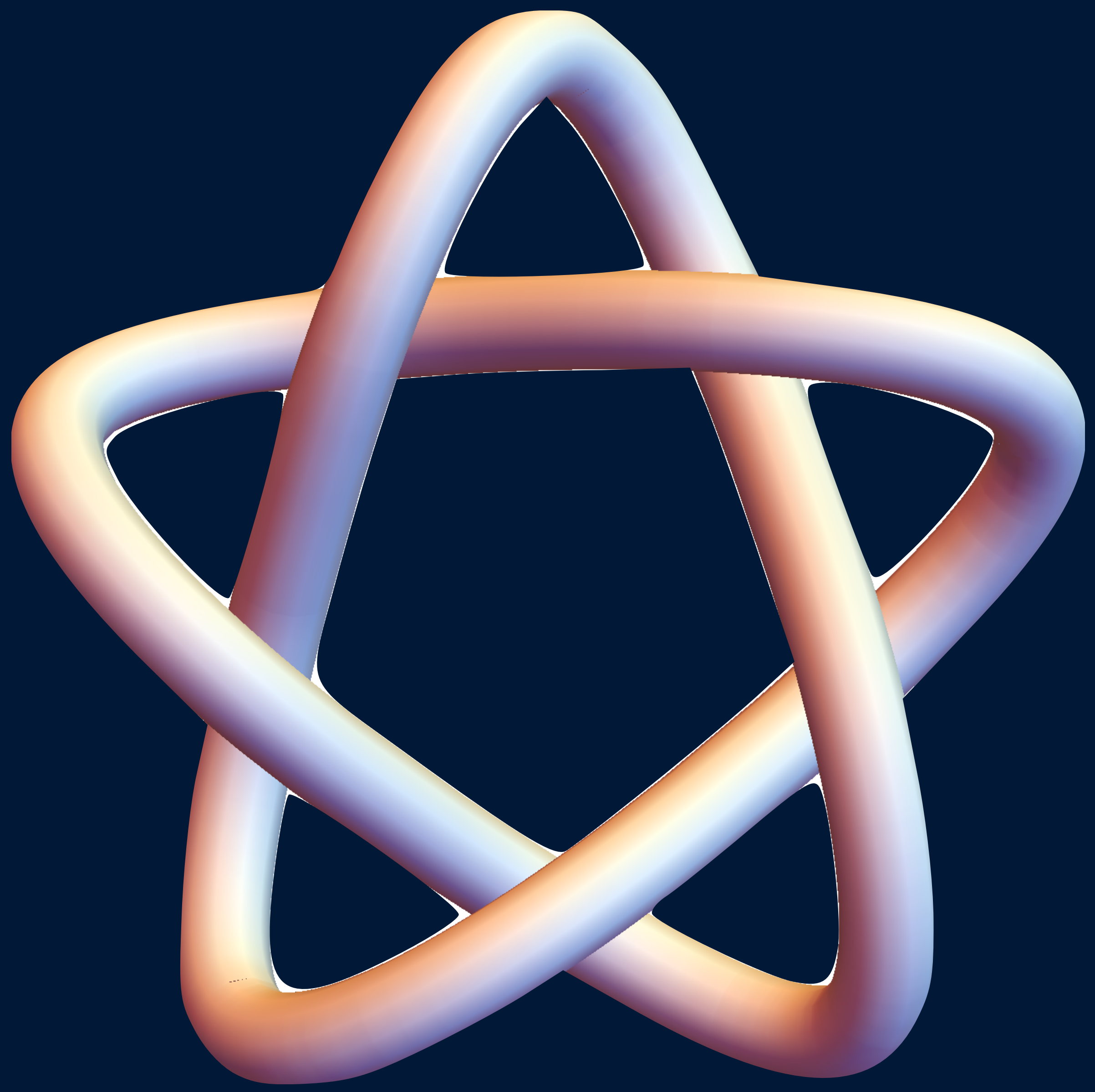
$$\Sigma \circ \Phi^{-1}(V(f)) = \text{Knot}$$



Braids

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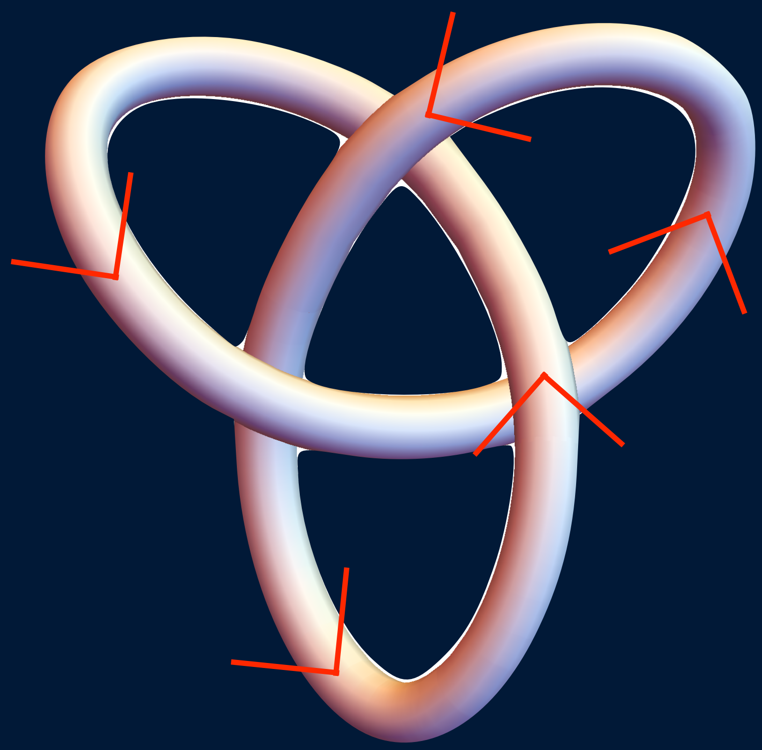

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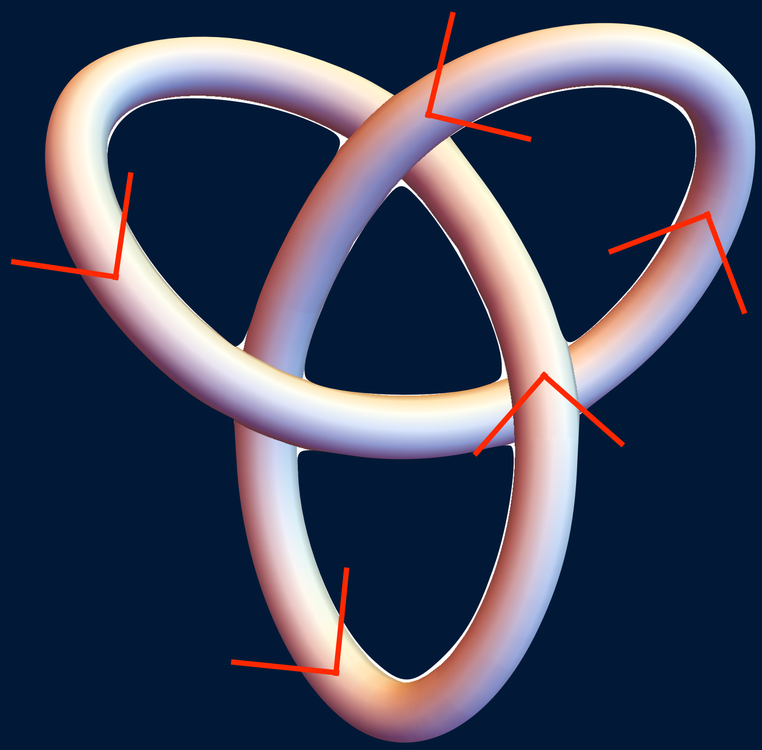

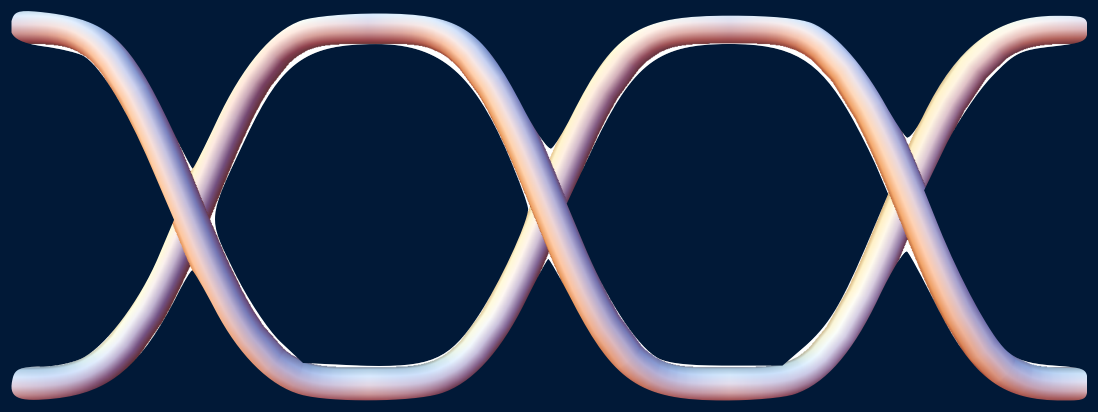

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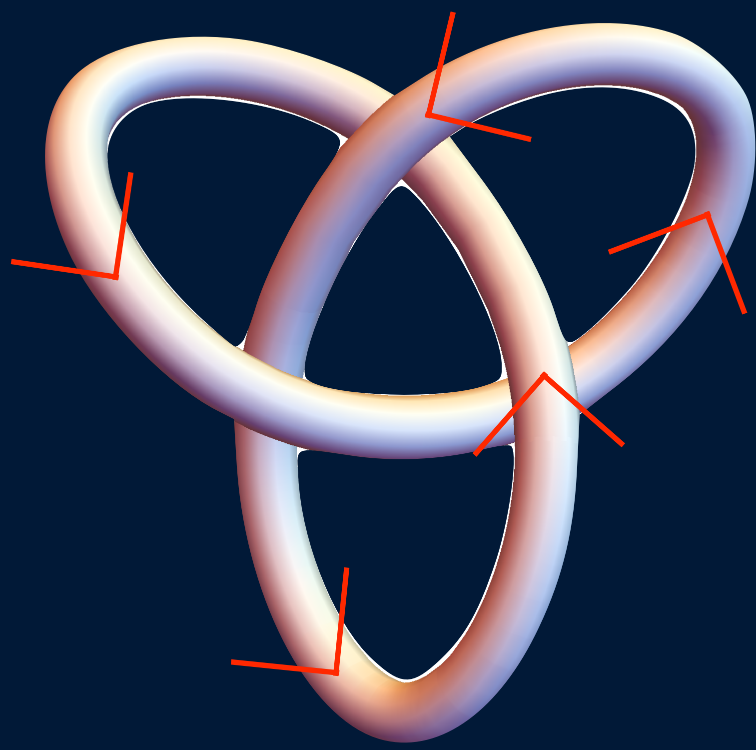

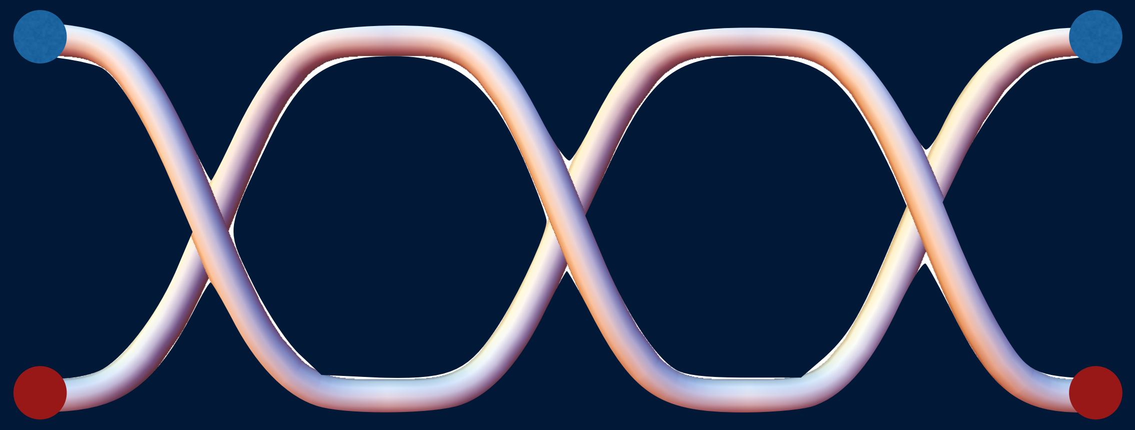

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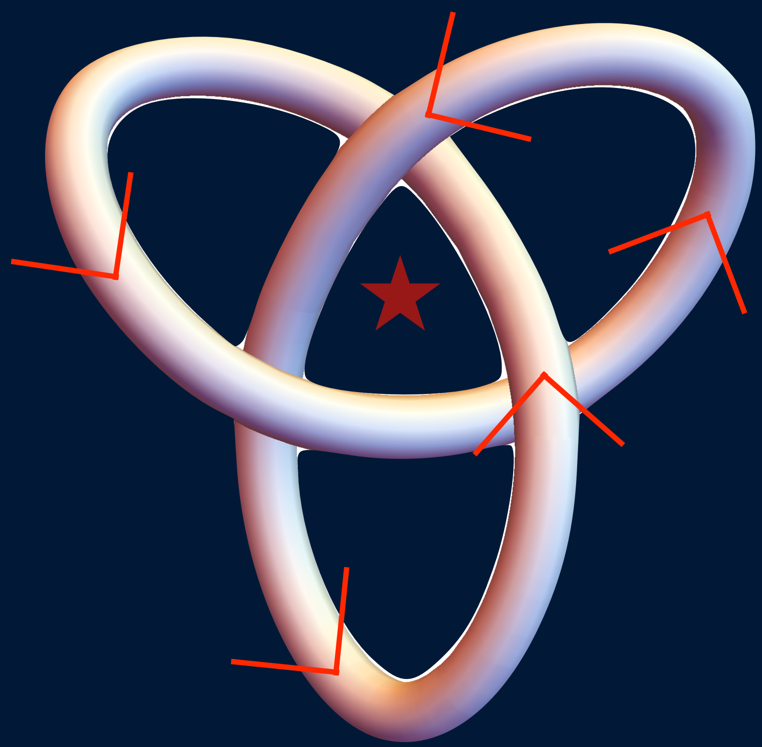

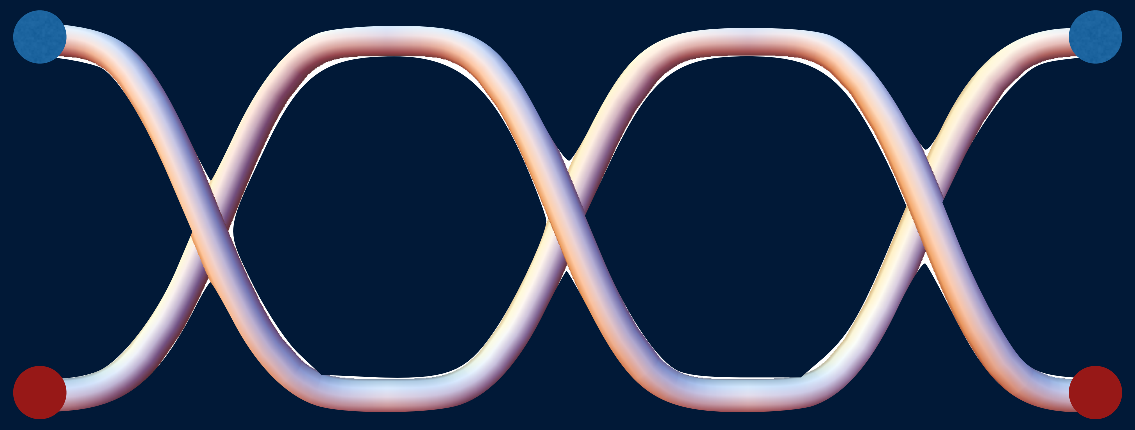

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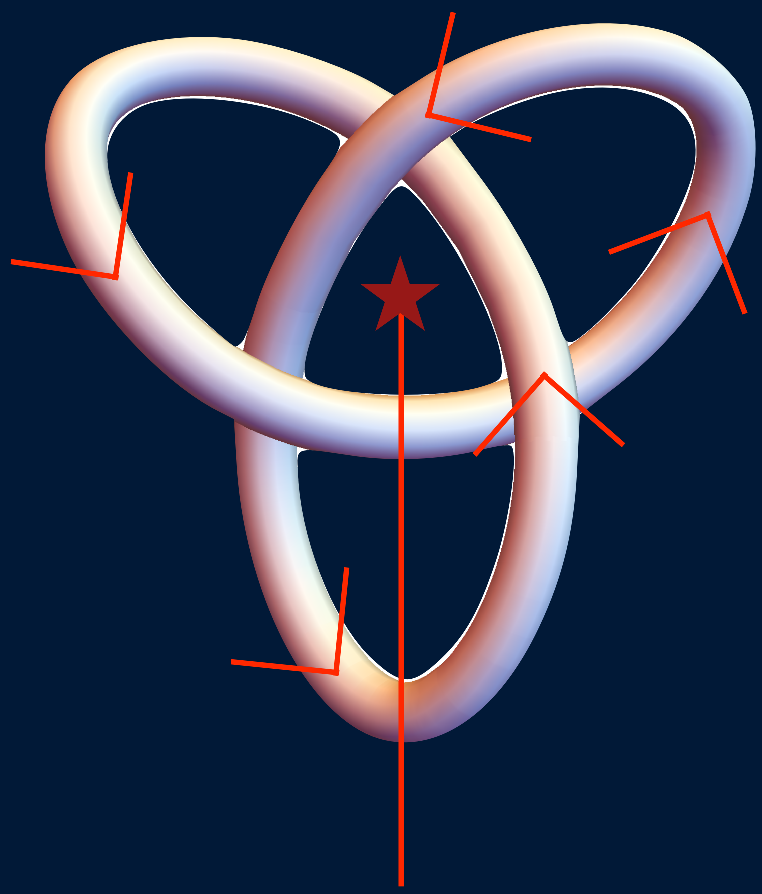

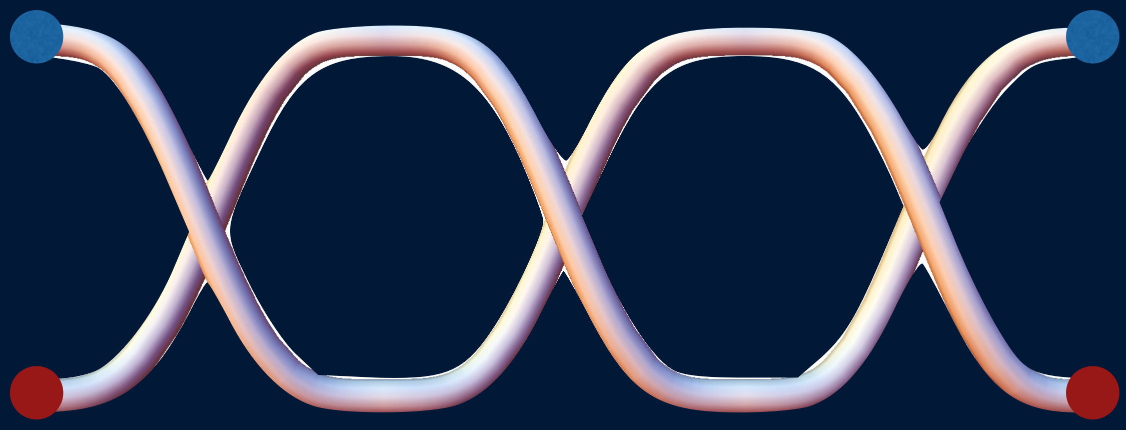

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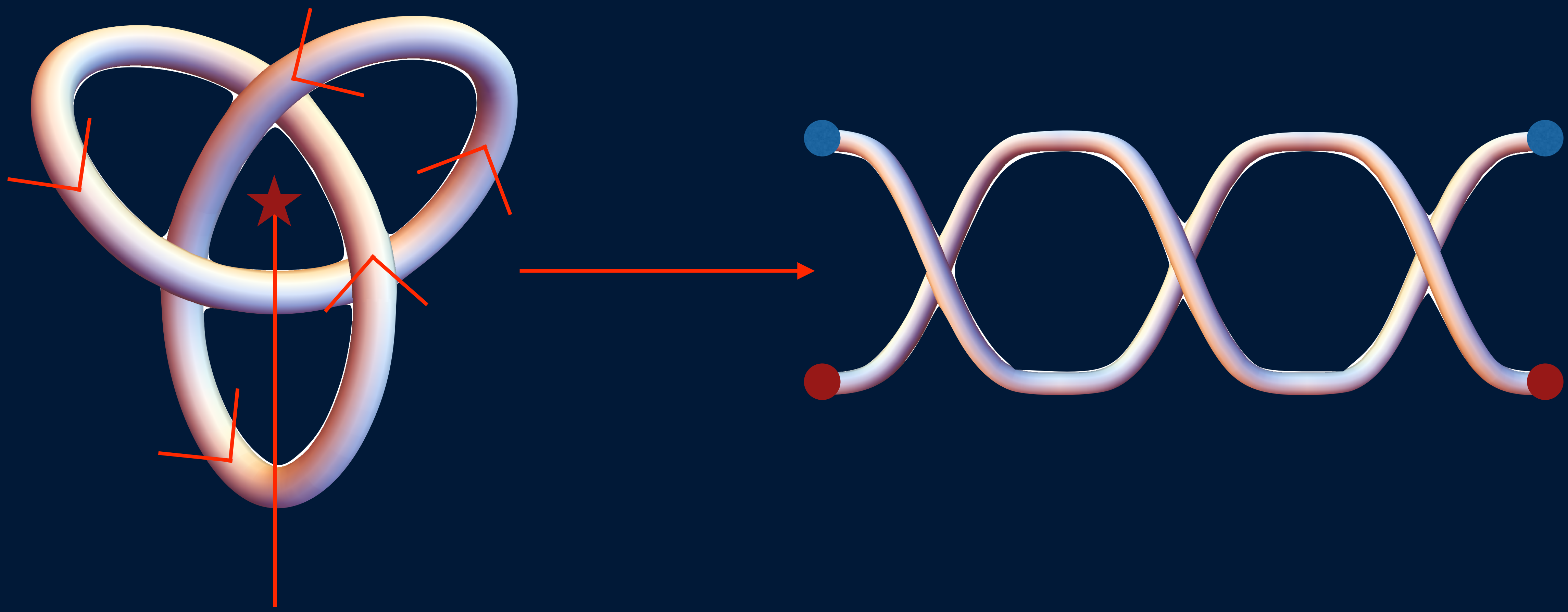

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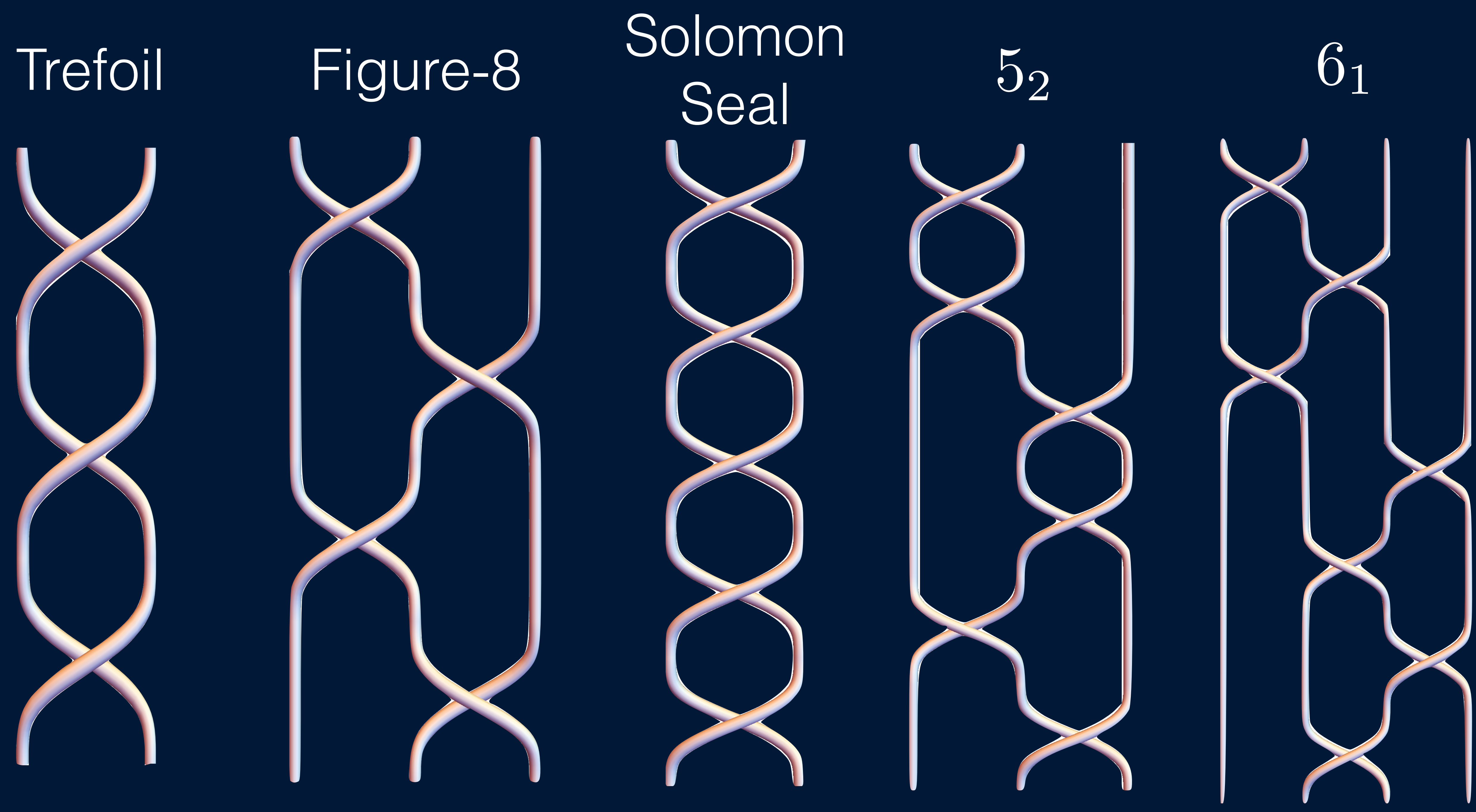
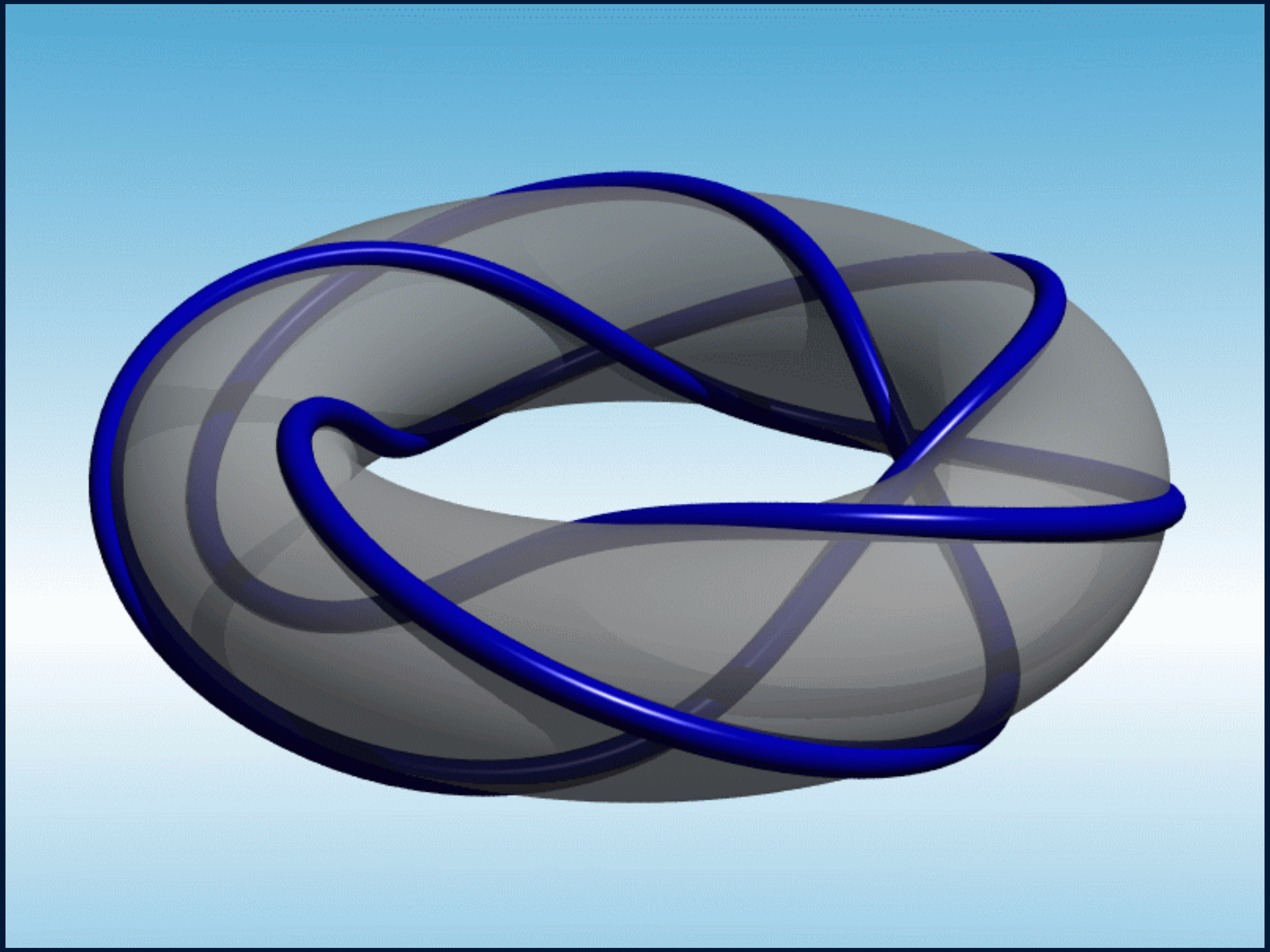
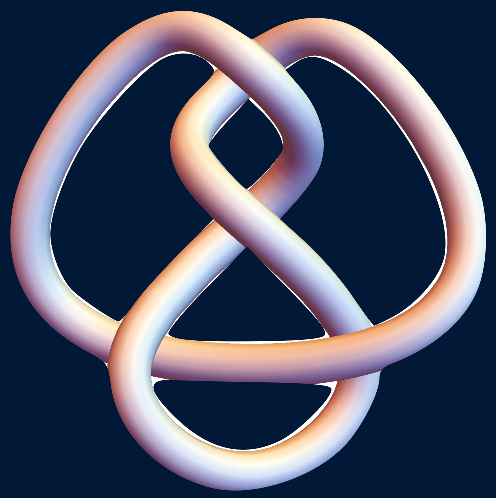


Torus Knots



$$f(u,v) = u^p - v^q$$

From Knots to Braids to Polynomials



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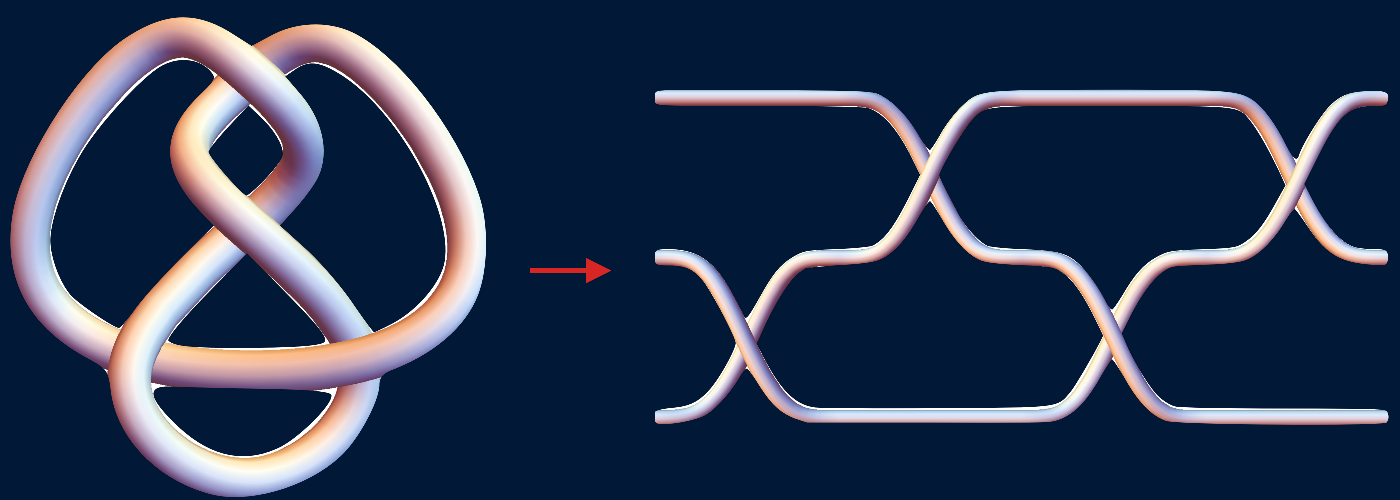

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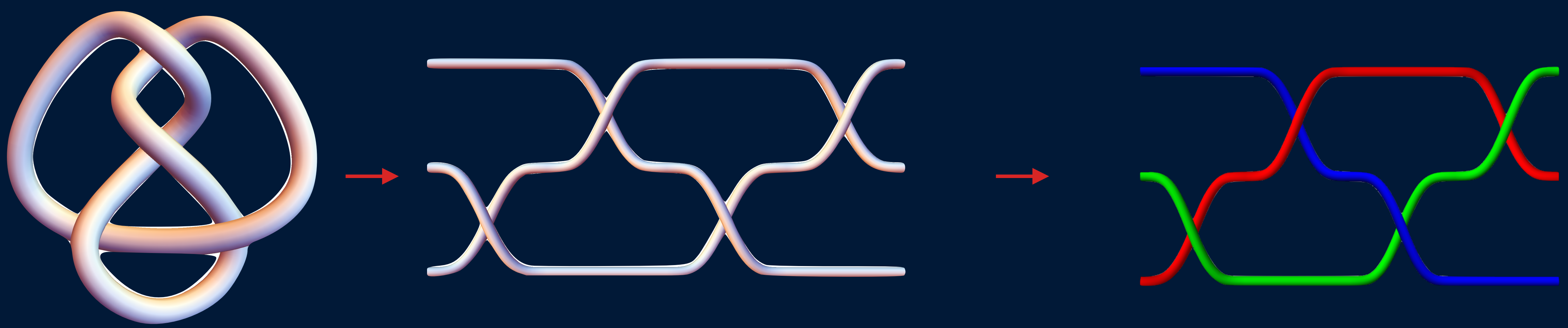

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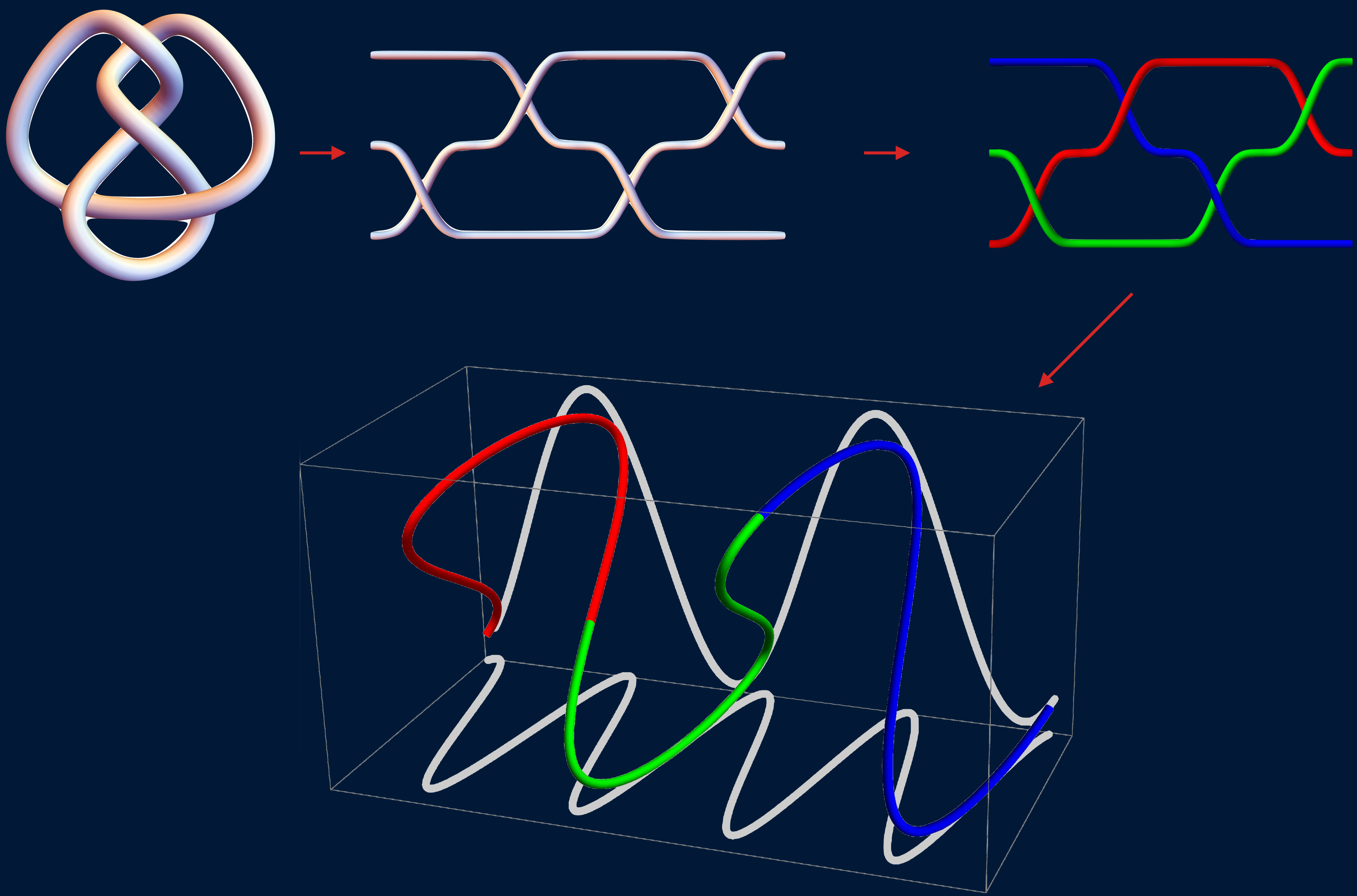

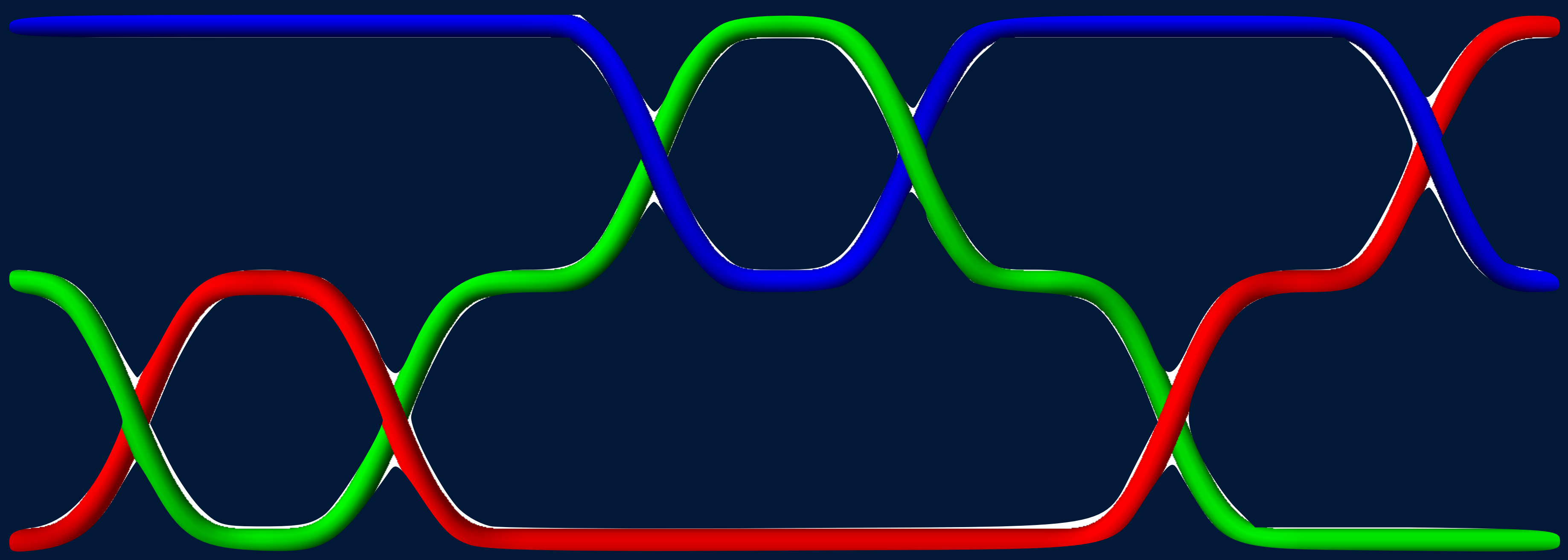

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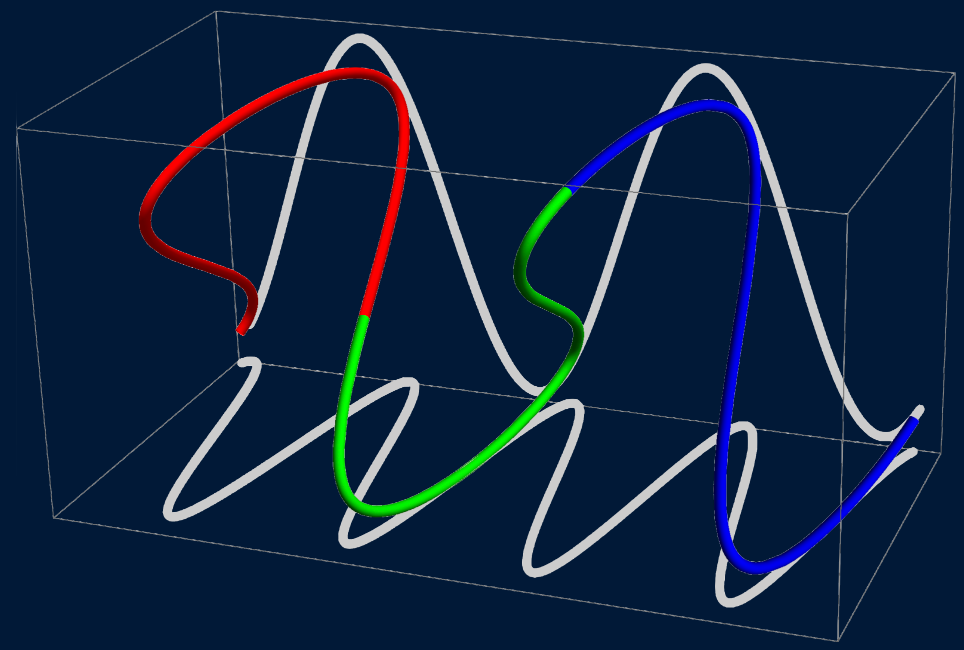

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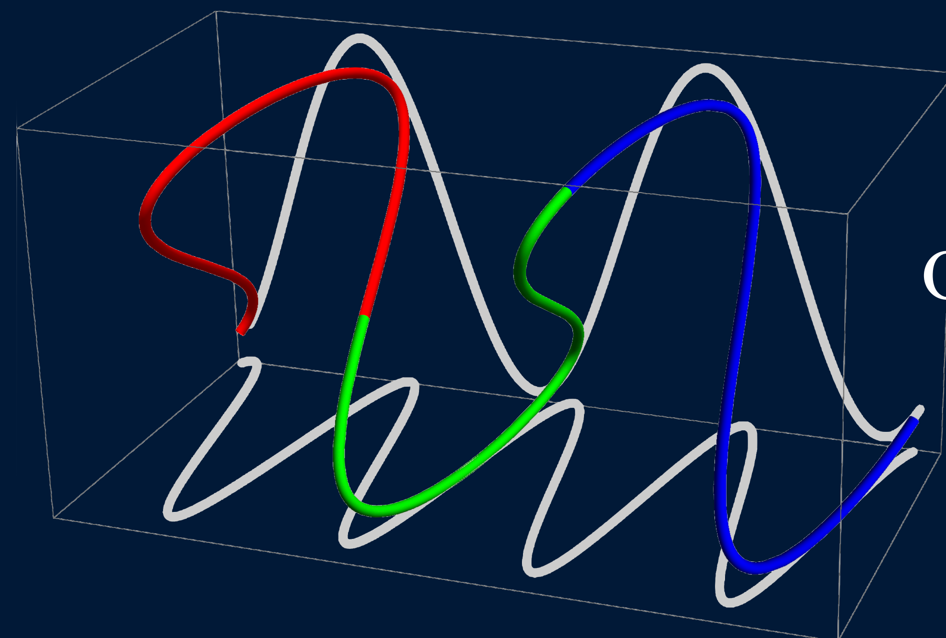
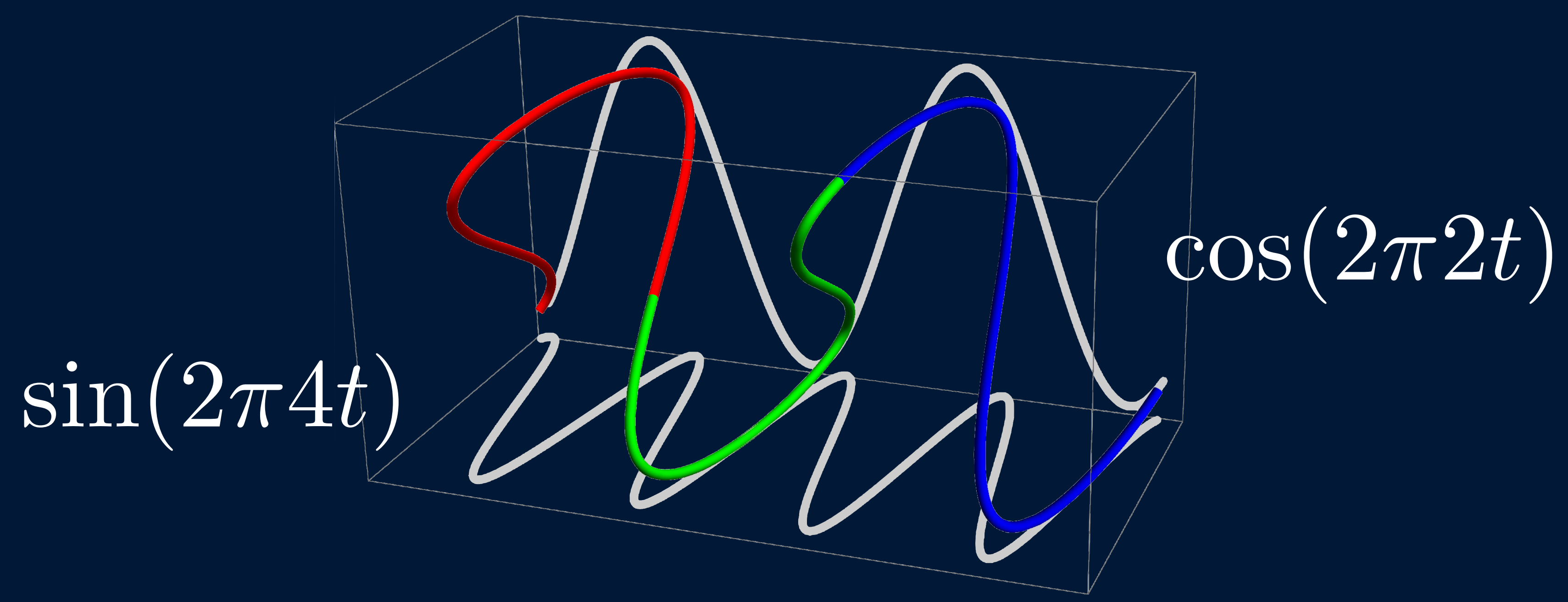


$\cos(2\pi 2t)$

From Knots to Braids to Polynomials



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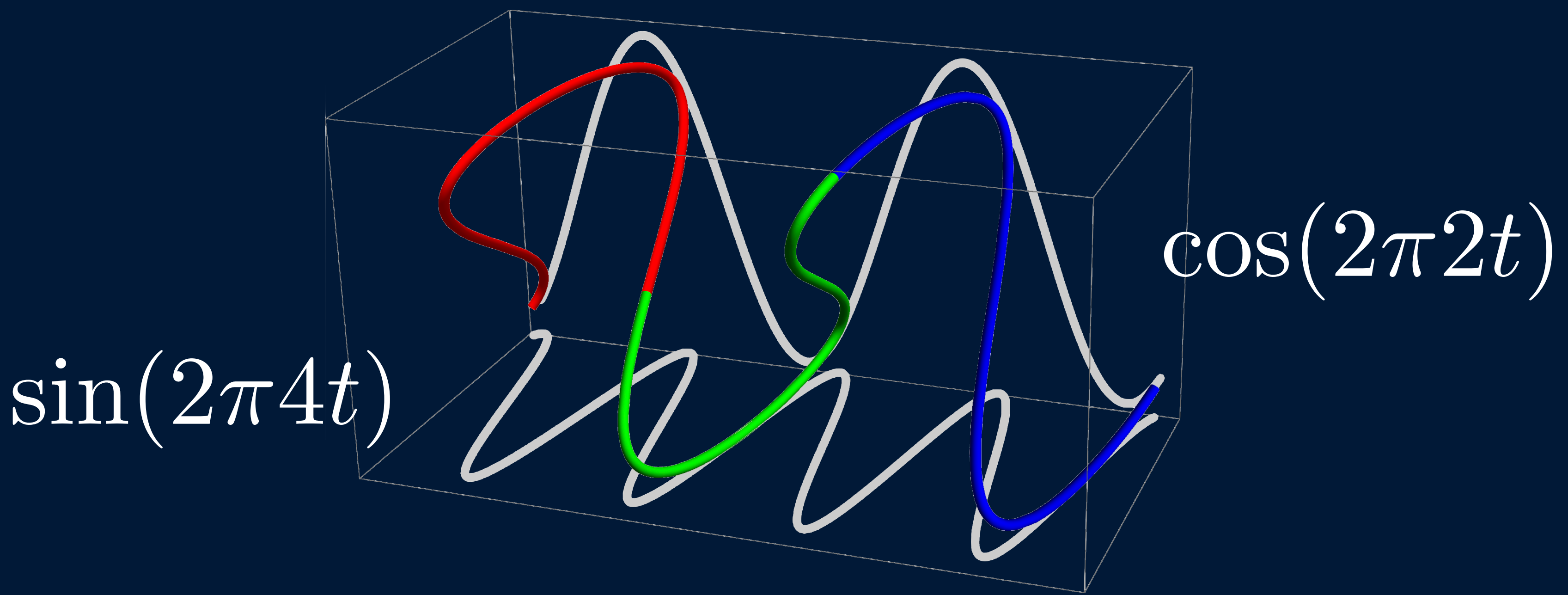
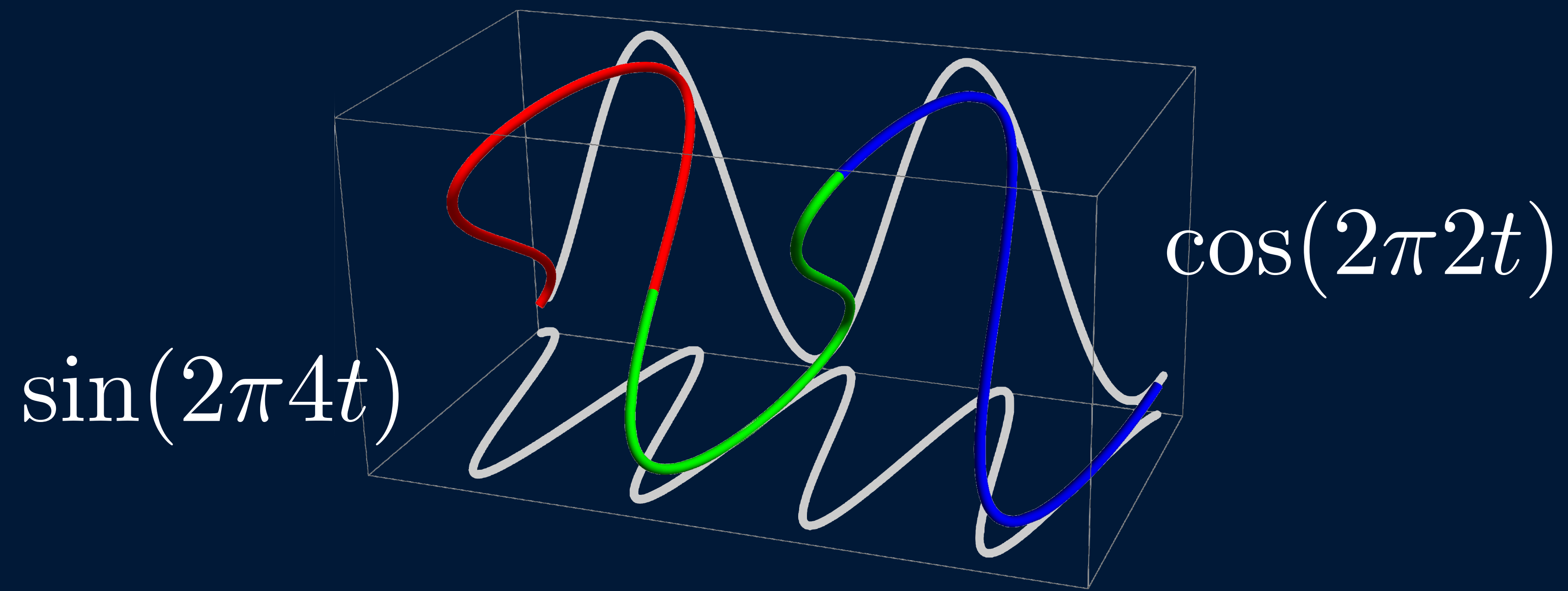


$$U(t) = \cos(2\pi 2t) + i\sin(2\pi 4t)$$

From Knots to Braids to Polynomials



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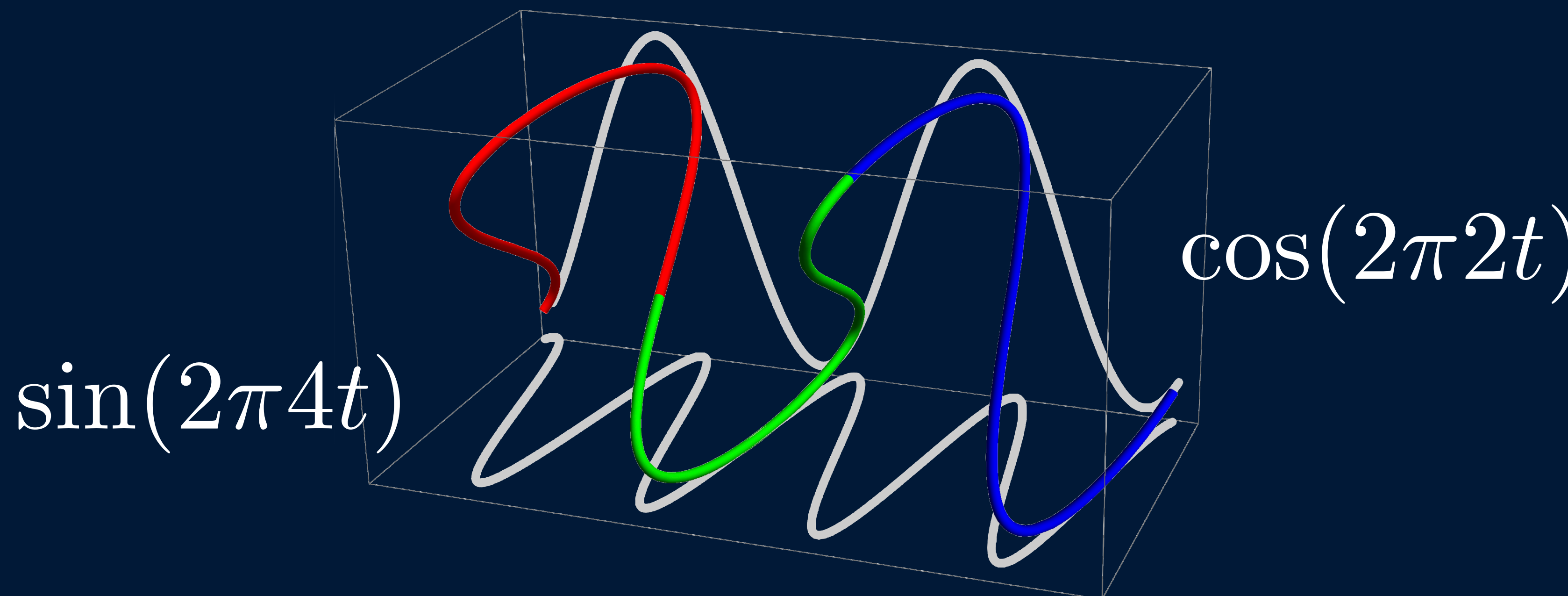
$$U(t) = \frac{1}{2}(v^2 + v^{-2} + v^4 - v^{-4}) \qquad \text{with } v = \exp(2\pi i t)$$

From Knots to Braids to Polynomials



$$U(t) = \cos(2\pi 2t) + i \sin(2\pi 4t)$$

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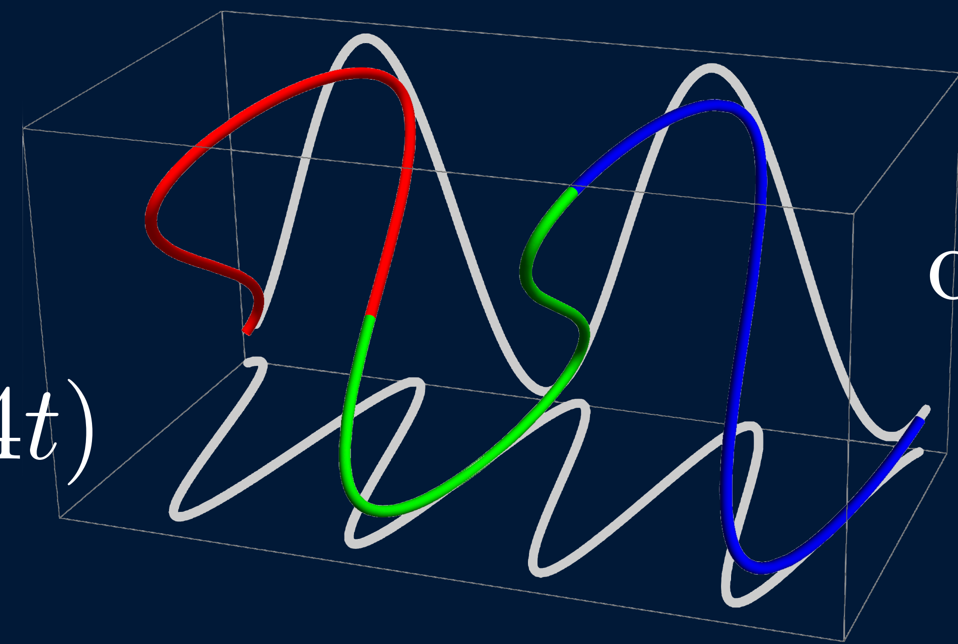
$$u_1(t) = U(\tfrac{t}{3}); \quad u_2(t) = U(\tfrac{t+1}{3}); \quad u_3(t) = U(\tfrac{t+2}{3})$$

From Knots to Braids to Polynomials



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$$p(u,v) = \prod_{i=1}^{3}(u - u_i(t))$$

Is this what we want?

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$$f(u, v) = u^2 - v^3$$

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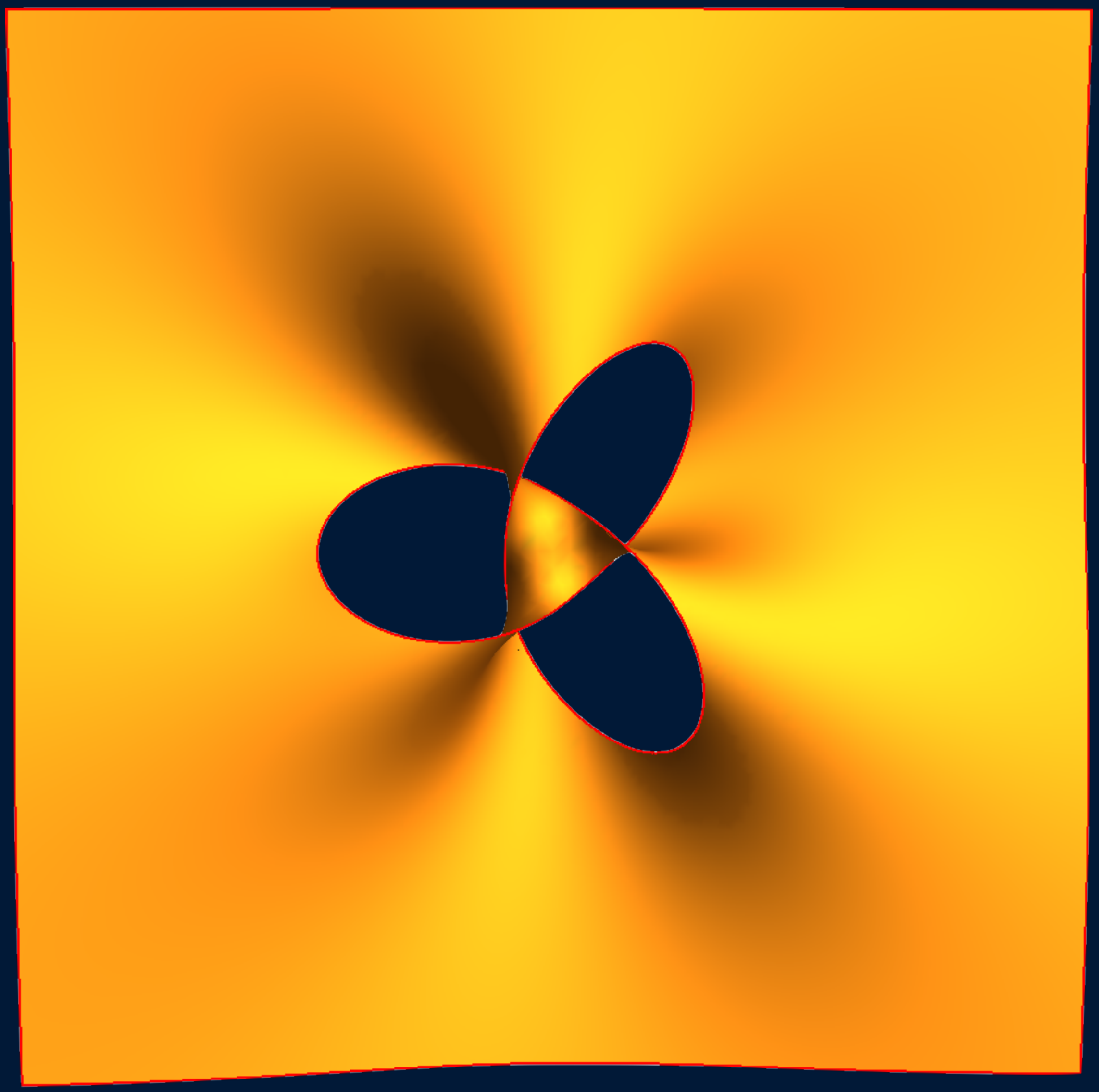
$$f(u, v) = u^2 - v^3$$



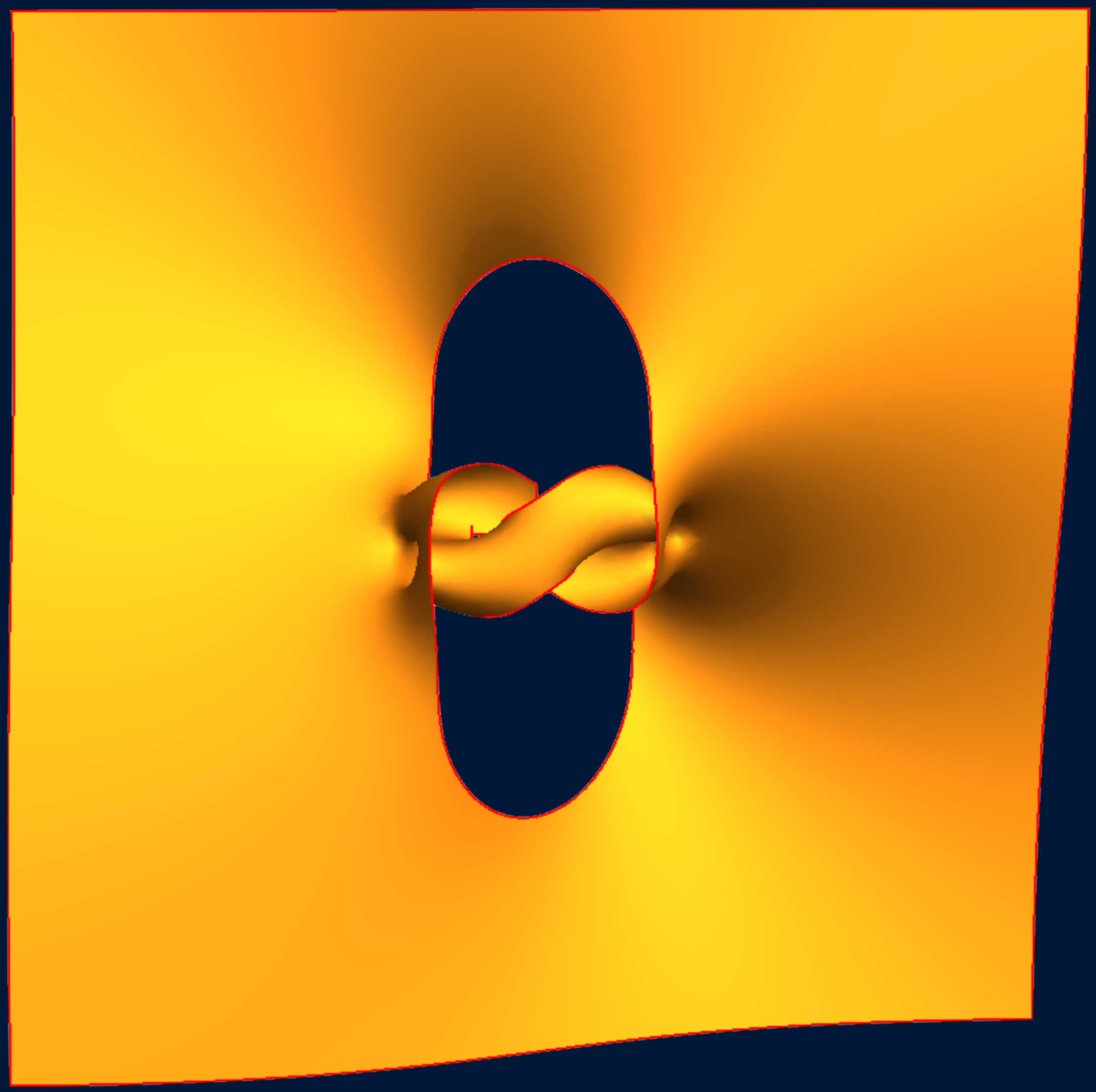
$$f(u, v) = \frac{3}{4}u\left(v^{*}\right)^{2} + \frac{\left(v^{*}\right)^{4}}{8} - \frac{\left(v^{*}\right)^{2}}{2} + u^{3} - \frac{3uv^{2}}{4} - \frac{v^{4}}{8} - \frac{v^{2}}{2}$$

Figure 8

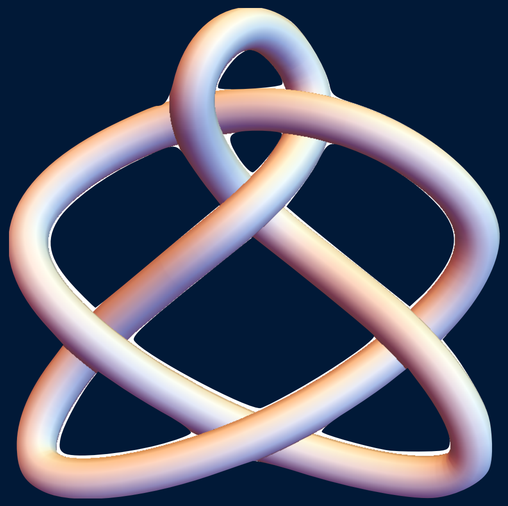
$$f(u, v) = \frac{3}{4} u \left(v^*\right)^2 + \frac{\left(v^*\right)^4}{8} - \frac{\left(v^*\right)^2}{2} + u^3 - \frac{3uv^2}{4} - \frac{v^4}{8} - \frac{v^2}{2}$$

Figure 8

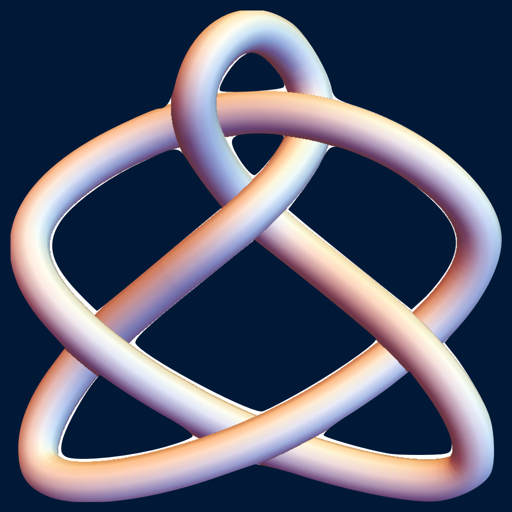
$$f(u,v) = \frac{3}{4}u\left(v^*\right)^2 + \frac{\left(v^*\right)^4}{8} - \frac{\left(v^*\right)^2}{2} + u^3 - \frac{3uv^2}{4} - \frac{v^4}{8} - \frac{v^2}{2}$$



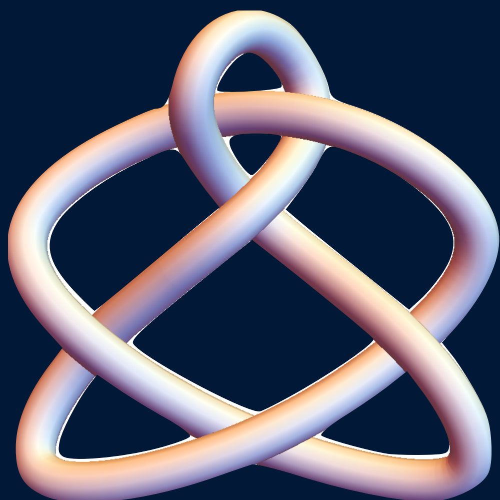
Miller Institute Knot



Miller Institute Knot



Miller Institute Knot



$$f(u,v) = 432 - 48u + u^3 - 1350v + 192uv - 9u^2v + 386v^2 - 45uv^2 - 2523v^3 + 237uv^3 + 1452v^4 - 120uv^4 + 19v^5 + 576v^6 - 512v^7 + 522v^* + 48uv^* - 9u^2v^* + 530(v^*)^2 - 45u(v^*)^2 - 939(v^*)^3 - 147u(v^*)^3 + 2604(v^*)^4 + 120u(v^*)^4 - 269(v^*)^5 + 576(v^*)^6 + 512(v^*)^7$$

Miller Institute Knot

Any questions?