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Spectral Geometry of Fuzzy Spaces

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Abstract

In this thesis, a number of tools are developed to better understand fuzzy spaces and finite noncommutative geometries in general. These tools depend only on the spectrum of the Dirac operator. Dimensional measures based on Weyl's law and heat kernel asymptotics are defined. A new dimensional measured called the spectral variance is defined as a modification of the spectral dimension to remove some of its undesirable properties. Volume measures based upon the Dixmier trace and the work of Stern are adapted to the finite setting and tested on the fuzzy spaces. The distance between two geometries is investigated by comparing the spectral zeta functions using the method of Cornelissen and Kontogeorgis. All of these tools are then used to investigate the fuzzy sphere, the fuzzy tori and the random fuzzy spaces introduced by Barrett and Glaser. The role of symmetry in the creation of fuzzy spaces is investigated using the characterisation of the Dirac operator given by Barrett. It is shown that all $SU(2)$-equivariant Dirac operators for type $(0,3)$ and $(1,3)$ fuzzy spaces produce the round metric on the sphere, despite the Dirac operators not agreeing with Grosse-Prešnadjer or Barrett operators. A pathway for further research is presented along these lines.

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CONTENTS

1

INTRODUCTION

Modern physics has many fundamental questions that require answering. Such as explaining the existence of dark matter and dark energy, explaining the matter-antimatter imbalance, why does spacetime have 3 spatial and 1 temporal directions? However arguably the most sought after answer is to the question of how to describe gravity quantum mechanically. The apparent incompatibility of gravity with that of quantum field theory is famously difficult to resolve. Birthing ideas such as string theory and loop quantum gravity as potential models, amongst others.

Gravity is best understood by Einstein's theory of general relativity [1]. In which the "force" of gravity is understood in a purely geometric manner - as the curvature of spacetime. This theory has been subjected to the most penetrating of tests over the last century or so and come out victorious every time. Recently the direct measurement of gravitational waves is just another feather in general relativity's cap [2]. The modern, mathematically rigorous, view of quantum field theory is described in a quite contrasting manner. By a process of assigning noncommutative algebras - an algebra of observables - to regions of spacetime [3, 4]. The best quantum field theory of nature, known as the standard model of particle physics, is also very successful. It describes the known fundamental building blocks of matter and their interactions near perfectly. However, one of its limitations is that it only accounts for three of the four known fundamental forces. It includes the strong and weak nuclear forces, and electromagnetism but does not included gravity.

Despite general relativity (GR) being a classical field theory akin to classical electromagnetism. Attempts to treat gravity as a quantum field theory fail to be useful. When converting electromagnetism to a quantum field theory the process of renormalisation is required. This process can be loosely understood as measuring a finite number of quantities to renormalise the theory with. For electromagnetism these are the electron mass and charge as measured at a given energy scale. If one does not undergo this process, the quantum theory of electromagnetism predicts infinite values for quantities known to be finite. However once sufficient renormalisation has occurred the theory is highly accurate. Applying the same process to gravity would require an infinite number of measurements to be made in order to renormalise the theory. This is impossible to do in practice and in such situations a theory is called non-renormalisable. Consequently, in situations where the energy scales are close to the Planck scale ($E_P \sim 10^{19} GeV$) the theory fails to be predictive and requires us to consider new approaches. Note that for energies much lower than the Planck scale, the quantum theory of general relativity can still be predictive and can provide potential quantum corrections to the classical field theory results [5].

If a high-energy quantum theory ($E > E_P$) of gravity is found, that reduces to general relativity for low energies, it is called a UV completion. The UV completion does have to be a quantum field theory and it may not be unique[1]. The search for such a theory has produced many different potential quantum gravity models. The most famous of these is string theory, which is an extension of quantum field theory where the 0-dimensional point-like particles, are replaced by 1-dimensional string-like objects. The different matter and force-carrying particles are formed by the different vibrational modes of the fundamental string. Gravity is included in this description as one of the vibrational modes is always the graviton - the hypothetical particle that conveys the force of gravity. The cost of this approach is the inclusion of six extra dimensions to the usual 4-dimensional spacetime [6].

1 The Schrödinger equation has both the Dirac equation and the Klein-Gordon equation as possible UV completions.

There are still many questions to be answered within string theory. The theory is defined on a fixed background - called its vacuum state. This means that the theory is not background independent, which is a core assumption of general relativity. Also, the number of different consistent vacuum states of the theory is very large due to the numerous ways to *compactify* the extra six dimensions. With the number originally being estimated at 10^{500} [7] and with modern estimates of $10^{270,000}$ [8]. As such, to make predictions with string theory requires arbitrarily picking a vacuum and hoping it is the correct setup. This problem is called the landscape problem of string theory, see [9] for a recent review. This amongst other issues such as the necessary existence of supersymmetric particles, for which no evidence exists, has lead to increased interest in other approaches.

Many of the other attempts to find a UV complete theory of quantum gravity, are based around making spacetime have a discrete nature at the smallest scales. The reason for this is to define a path integral over geometries that is free of divergences. The path integral for GR is

$$Z = \int Dg e^{iS_{EH}[g]}, \tag{1}$$

where $S_{EH} = \int \frac{c^4}{16\pi G}(R - 2\Lambda)\sqrt{-g}d^4x$ is the Einstein-Hilbert action and Dg is an integration measure on the space of all Lorentzian metrics. This integration measure is poorly defined and attempts to quantise this path integral lead to the non-renormalisable quantum field theory mentioned above. Making spacetime discrete yields two main benefits to this effect. The first is that the integral over all geometries is often turned into a combinatorial problem, making the integral well defined. The second is that in the quantum field theory of GR, the divergences occur from the high energy aspects of the theory. A consequence of many discretisation procedures is that there becomes a maximum allowed energy state. Such a maximum allowed energy is called a UV cutoff and is usually taken to coincide with the Planck energy E_P.

Covariant loop quantum gravity (LQG) [10], causal dynamical triangulations (CDT) [11, 12, 13] and causal set theory [14, 15] are all models that discretise spacetime,

where different aspects of spacetime are treated as fundamental. In LQG the background independence of general relativity is made sacred, whereas in CDT and causal sets the causal nature of events is the founding pillar.

LQG and CDT find their origin in Regge calculus [16]. Originally developed as a tool for numerical relativity, Regge calculus builds a discrete version of spacetime out of simplices. The curvature of the spacetime is calculated by the deficit angle when gluing simplices together. A key component of Regge calculus is a discrete version of the Einstein-Hilbert action. This allows one to define a path integral over these discrete geometries. The edge lengths of the simplices are allowed to vary in the full Regge action, leading to a complex action. Analytic calculations are extremely difficult with this action however, numerical simulations of Euclidean spaces have been conducted, for instance see [17, 18]. Note that because the edge lengths are allowed to be any size, this does not incorporate any quantum features into the theory. This is the starting point for the development of LQG and CDT.

CDT makes a simplification to the action by making all the edge lengths the same, which greatly reduces the complexity. This automatically provides us with a so-called atom of geometry and builds into the theory a fundamental smallest length scale, called a Planck length. Lorentzian versions were developed in [12]. Upon numerical exploration, three distinct phases of the theory were found, depending on the values of different parameters in the action. One of these phases produces geometries that undergo a dimensional reduction. They have a dimension of 4 at low energies, and a dimension of 2 at high energies. This is notably different to the scenario in string theory, where the extra dimensions are accessible at high-energies and hidden at low-energies.

LQG describes gravity using spin-foams. Spin foams can be conceptually viewed as the evolution of spin-networks, a discrete model of $3D$ Euclidean space. A spin-network of 3-dimensional space is built from simplices, which are tetrahedra in this dimension, like in Regge calculus. Each of the 6 edges of the tetrahedra are labelled with a spin representation of $SU(2)$, denoted j_i where i labels each edge. These representations can be viewed as labelling the lengths of each edge. The use of

$SU(2)$ representations implements a Planck length in this theory, due to the fact that j_i have to take half-integer values $\{1/2, 3/2, 5/2, \dots\}$. Also as the representations are assigned to each edge, they are subject to conditions to make the tetrahedra close. These closure conditions are distilled in the form of assigning a Wigner-6j symbol to each tetrahedra. The area and volume of the tetrahedra are also quantised by this procedure, see [10] for details. Applying this procedure to the entire triangulation of a space produces as spin-network. Spin-foams can be defined independently of spin-networks by implementing a similar procedure for a 4-simplex [19].

The action for the $3D$ Euclidean model is produced by integrating over all possible spin configurations of all the tetrahedra. It can be shown to produces the $3D$ Einstein-Hilbert action in the semiclassical limit of $j \to \infty$ [20, 21]. The situation is much harder for the Lorentzian $4D$ case, and such a semiclasssial limit has not been formally shown for spin foams. Despite this, the classical values for black-hole entropy, Unruh temperature and Hawking radiation can be derived from the LQG approach [22, 23, 24].

Causal set theory abstracts even further, building spacetime from partially ordered sets. A random set of points is generated and a causal relationship between them is defined using a certain type of partial order. A causal structure is only available in Lorentzian signature so if causal sets are manifold discretisations, they are of Lorentzian manifolds. The set of points can be sampled from a Lorentzian manifold along with their causal relations to each other, or they can be randomly generated and one could attempt to map these into a Lorentzian manifold, whilst preserving the causal relations. There is a causal set analogue of the Einstein-Hilbert action called the Benincasa-Dowker action [25], for which numerical simulations have been investigated for low dimensions [26, 27, 28]. Evidence of manifold-like phases of the theory were discovered. Also, a prediction for the cosmological constant is possible that fluctuates around zero. See [15] for a review of recent causal sets.

Each of these approaches have a fundamental quanta of geometry, an atom of spacetime. The quantum effects are due to the presence of such atoms. The dynamics of gravity is then how these atoms of spacetime interact. The dynamics is

governed by a path integral over all allowed geometries. Where each geometry is weighted by a factor of $\exp(iS)$, for some action S that recovers the Einstein-Hilbert action of GR in some classical limit. The inclusion of the rest of physics in LQG, CDT and causal sets is not obvious and a major area of research. As ideally, the frameworks would also neatly explain the other aspects of nature, by either including the standard model of particle physics or by another unknown process. If they do not, then serious questions about the usefulness of such models are drawn into question. How can gravity and matter interact if they are not expressed in the same framework?

The approach of investigation in this thesis is via noncommutative geometry. Understanding quantum gravity from noncommutative geometry begins from a different viewpoint than other theories. It begins by noticing that the transition from classical mechanics to quantum mechanics is hinged on the idea of noncommutativity. Classical mechanics is understood in terms of differential geometry. The position and momentum of particles are functions of the phasespace, as such they are commuting variables. You can know both quantities simultaneously at any instant of the theory. The key revelation of early quantum mechanics was the introduction of noncommuting position, x, and momentum, p, by Heisenberg, to explain the quantisation of atomic spectra [29, 30, 31]. The position and momentum became infinite matrices, formally called operators, that satisfy the relation: $[x, p] = xp - px = i\hbar$. A consequence of this noncommutativity is the Heisenberg uncertainty principle, which states that the more accurately to you know the position, the less accurately you know the momentum. The modern view of quantum mechanics is built from these ideas, where the observable quantities of the universe, are expressed as Hermitian operators. These operators need not commute, and if they do not commute then there exists an uncertainty relation between them. Thus the observables in a quantum theory form a non-commuting algebra. The starting point of noncommutative geometry was to develop a common framework, on which the noncommutative algebras of quantum mechanics, and the differential geometry of classical mechanics, can be simultaneously expressed.

The mathematical study of differential geometry and noncommutative algebras has traditionally been distinct. However, there are various powerful correspondences between geometric and algebraic objects that link them. The most famous is that provided by Gelfand and Naimark (1943) [32] which allows us to pass between a locally compact topological space to a commutative C^* algebra.

$$\text{compact Hausdorff topological space} \quad \longleftrightarrow \quad \text{commutative unital } C^* \text{ algebras}$$

The commutative C^* algebra in the duality is non other than the space of continuous functions over that topological space M, $f : M \to \mathbb{C}$, equipped with pointwise multiplication and addition. Likewise any unital commutative C^* algebra can be viewed as the space of continuous functions over a compact Hausdorff space, $C(M)$. This provided a way to generalise the notion of a topological space in a manner that was not possible before, by studying *noncommutative* C^* algebras. The premise is that noncommutative C^* algebras can be treated as the algebra of functions on a *noncommutative topological space*. Many of the usual topological properties can be formulated in a purely algebraic manner. Such as compactness, which is provided by making the algebras unital, and connectedness which is given by making the C^* algebra projection-less.

The existence of this duality raises the question - what other geometric quantities can be translated into a purely algebraic way? It turns out that there are many dualities between geometric constructions and algebraic constructions. Such as the Serre-Swan theorem [33, 34]. Which states that vector bundles over a compact topological space M are equivalent to finitely generated projective modules over the commutative C^* algebras $C(M)$. By studying finitely generated projective modules over noncommutative C^* algebras, you are in effect studying vector bundles over noncommutative spaces. The general area of study of generalising geometric constructions to noncommutative algebra is broadly known as *noncommutative geometry* (NCG).

Noncommutative geometry is a fascinating field of study for both mathematicians and theoretical physicists alike. For mathematicians it offers unexplored mathematics and a new viewpoint to tackle problems from. The realisation that all of topology and differential geometry may in-fact be a very specific case of a much wider realm, is the sort of realisation that produces leaps and bounds in understanding.

For physicists, the world of noncommutative geometry also offers a new view on some of the most well tested theories of reality. Given that quantum field theory is described in terms of noncommutative algebra, a natural question is whether one can pass from the pseudo-Riemannian geometry description of general relativity to a purely algebraic description using the tools of noncommutative geometry? In doing so, is a single framework developed that describes both quantum field theory and general relativity in a compatible way? This is the focus of a lot of current research and there is a construction to pass between compact Riemannian spin manifolds to a purely algebraic description which was recently formally shown to be a duality [35].

The fundamental ingredients of this duality is a spectral triple, $(\mathcal{A}, \mathcal{H}, D)$, the details of which are outlined in section 2.2.1. For a compact Riemannian spin manifold these consist of the algebra of smooth functions on the manifold M, $\mathcal{A} = C^\infty(M)$, the Hilbert space of square integrable spinor sections, $\mathcal{H} = L^2(S)$, which carries an action of the algebra, and the Dirac operator $D = \not{D}$.

The Dirac operator $\not{D} = -i\gamma^a e_a^\mu \nabla_\mu^S$ encodes the metric via the veilbeins e_a^μ, which satisfy the relation $\delta^{ab} e_a^\mu e_b^\nu = g^{\mu\nu}$. There are a number of different actions that are defined in terms of veilbeins that produce to Einstein field equations as their equations of motion. Such as the Palatini action [36], the Plebanski action [37] and the Holst action [38]. Therefore it is plausible that in principal a path integral exists in terms of the Dirac operators that has the same content as the GR path-integral eq. (1). The spectral action [39] does precisely this. Defined as

$$S(D) = Tr(f(D/\Omega)), \tag{2}$$

where f is a smooth approximation of a cutoff function that regularises the trace, and Ω is an energy scaling factor that can be used in the renormalisation group equations [40]. This action has the following asymptotic formula:

$$Tr(f[D/\Omega]) \overset{\Omega\to\infty}{\sim} \overbrace{f_2\Omega^2\frac{96}{24\pi^2}\int R\sqrt{g}d^4x}^{\text{Einstein-Hilbert}} + \overbrace{f_4\Omega^4\frac{48}{\pi^2}\int\sqrt{g}d^4x}^{\text{Cosmological Constant}} + \frac{f_0}{10\pi^2}\int(\frac{11}{6}\underbrace{R^*R^*}_{\text{Gauss-Bonnet}} - \underbrace{3C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}_{\text{Conformal gravity}})\sqrt{g}d^4x, \quad (3)$$

where $f_i = \int_0^\infty f(v)v^{i-1}dv$ are the moments of the cutoff function. These moments can be tuned along with Ω to select the Einstein-Hilbert action (or some modifications as shown in eq. (3)). For further details on how to use the spectral action and noncommutative geometry in cosmology see [40].

One of the most surprising success of this approach of noncommutative geometry, is the description of the standard model of particle physics as a spectral triple, that is minimally coupled to gravity and complete with neutrino-mixing [41, 42]. This model can be shown to produce a reasonable Higgs mass (with the original model famously over shooting with a value of $\sim 170GeV$) providing that all the fields in the model are deemed necessary - specifically by treating a coupling constant as a scalar field [43, 44]. These models are examples of *almost-commutative* real spectral triples. Where the commutative real spectral triple for an ordinary spin manifold, $(C^\infty(M), L^2(S), \not{D})$, equipped with a real structure[2] is combined with a finite-dimensional real noncommutative spectral triple which are defined later in section 2.2.6. For the standard model the algebra is taken to be $A = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$ and the Hilbert space is taken to be $H = \mathbb{C}^{96}$. For the Hilbert space a copy of $\mathbb{C}$ is taken for each left and right handed fermion and anti-fermion, accounting for the three generations. The Dirac operator is then given by the Yukawa coupling constants and the Majorana masses [3].

2 The concept of real structure is introduced in section 2.2. It can be viewed as a geometrical form of charge conjugation.

3 There is also a real structure (defined later in section 2.2) that interchanges the respective particles and anti-particles and there is a $\mathbb{Z}_2$-grading that distinguished the left and right handed particles.

The motivation behind this thesis is then asking the question, what happens if the commutative spectral triple of spacetime is replaced with one that is noncommutative? Would this amount to using noncommutative geometry to describe a theory of quantum gravity? The program is to investigate the use of spectral triples to describe spacetimes that have a suitable notion of a Planck length, that is compatible with the continuous symmetries of ordinary physics. This minimal length scale is viewed as there existing a maximum energy scale available and not a restriction on the measurements capable on the space. By limiting the highest energy, the divergences in the non-renormalisable quantum field theory of general relativity would be regularised.

One way to impose this physical wish list in NCG is by using a finite noncommutative real spectral triple to model spacetime. These are noncommutative generalisations of the spectral triples in the duality outlined by Connes [35]. As they have a noncommutative algebra of functions, the notion of a infinitesimal point is impossible on these spaces. You can think of this as not being able to simultaneously diagonalise the coordinate operators as they are now noncommutative. These spaces also have a bounded Dirac operator, which encodes the energy cut-off condition. This thesis will be concerned with *fuzzy spaces* which are finite real spectral triples where the algebra of function is taken to be a finite dimensional matrix algebra. The motivations for this is that matrix algebras are the simplest noncommutative algebras, and there exist fuzzy spaces that approximate the 2-sphere and the 2-torus. The construction of these are introduced in section 2.3 The fuzzy sphere has long been studied and the descriptions as a finite noncommutative real spectral triple were outlined in [45, 46]. These are finite noncommutative real spectral triple that still possess the Lie group symmetry of rotations by $SO(3)$. The Dirac operator for the fuzzy sphere has a spectrum which is just a truncation of the continuum spectrum. Thus in terms of spectral geometry, this makes the fuzzy sphere a near-perfect approximation of 2-sphere.

The spectral triple for the fuzzy torus was introduced in [47] and further studied in [48]. Many of the geometrical construction on the ordinary torus are shown to

have noncommutative analogues on the fuzzy torus. The fuzzy torus also possesses an analogous group symmetry to that of the ordinary 2-torus. The spectrum of the Dirac operator for the fuzzy torus was also shown to converge to the ordinary spectrum as the matrix size is sent to infinity. Thus fuzzy spaces can be viewed as approximations to ordinary manifolds, where the algebra of functions is deformed to be noncommutative.

It should be noted that the term "fuzzy space" does not always have the same meaning in the literature. The only common feature amongst the different definitions is the use of matrix algebras as noncommutative geometries. The presence of extra structures, such as Laplacians, Dirac operators or real structures are often not considered. For instance, the fuzzy sphere as introduced by Madore [49] is a noncommutative geometry in the general sense. As the algebra of functions are built from matrix representations of $\mathfrak{so}(3)$ and so is noncommutative. However, it does not fit into the spectral triple characterisation of Riemannian manifolds presented by Connes [50]. It lacked the notion of a noncommutative Dirac operator, which was provided later by Grosse and Presnadjer [51].

There are also slightly different definitions of the term "spectral triple" within the literature. A spectral triple here will be taken to mean a real spectral triple that satisfies the axioms laid out in [50]. It should be noted that there exist many finite noncommutative analogues of geometries outside of these strict axioms, see for example [52, 53, 54, 55, 56, 57, 58, 59, 60].

The possible Dirac operators on a fuzzy space were given an explicit expression in terms of Hermitian and anti-Hermitian matrices in [46]. This parametrisation allowed for the entries of the Dirac operator to be randomised, whilst keeping the algebra and Hilbert space fixed. The space of fuzzy Dirac operators forms a $\mathbb{R}$-vector space and so integration over all possible fuzzy Dirac operators is well defined. A Monte-Carlo simulation of the path integral over fuzzy spaces was conducted in [61], where the simplest actions were investigated with surprising results. The findings showed that the Dirac operators from certain models exhibit a phase transition upon varying the action parameter. It was found that near this phase transition,

the eigenvalues of the Dirac operators were similarly distributed to those of an ordinary manifold. However the analysis in [61] was qualitative and more concrete tools of comparison were needed. This is what chapter 3 of this thesis is concerned with.

As the only variable in the random fuzzy spaces is the Dirac operator, the tools developed are spectral in nature. That is they depend solely on the eigenvalues of the operator. Thus many of the tools are adapted from those used in the spectral geometry of manifolds. A number of different spectral measurements are introduced with the aim to define a notion of *dimension* and *volume* for a fuzzy space.

Using the fuzzy sphere as a key example of an ideal fuzzy space, the notion of symmetries in fuzzy geometry is investigated. Given that the fuzzy sphere and fuzzy torus both possess Lie group symmetries which preserve their structure. The precise nature of how these symmetries manifest in the Dirac operator is investigated. First for the commutative case and then for the fuzzy case. This is the subject of chapter 4.

This thesis is therefore split into three main chapters (not including this introduction).

Chapter 2 introduces the spectral triple formalism that the rest of the thesis relies on. In the first section, a constructive approach to the spin geometry of Riemannian manifolds is given. Specifically the Hilbert space of spinors and the Dirac operator for a Riemannian spin manifold are described in a constructive way. This is done by introducing the notion of a Clifford algebra bundle over a Riemannian manifold (M, g). The notion of a spin-c structure is briefly described in terms of the existence of a bundle isomorphism. The spin structure is defined as the existence of an anti-linear operator which plays the role of the real structure in a spectral triple. Whether these structures exist for all manifolds is not addressed, as it is a well studied problem, see [62] for instance. The Dirac operator is described in terms of local coordinates and how to reconstruct the metric from the Dirac operator is briefly outlined.

The next section gives the axioms for a spectral triple. An explanation of how some of the axioms are realised in the commutative setting is provided. The axioms

for a finite real spectral triple are given in full, along with any necessary algebraic definitions. A detailed account of Clifford algebras and Clifford modules is given. With the procedure on how to construct the high dimensional Clifford modules from lower dimensional ones explicitly demonstrated. The characterisation of the Dirac operator of a finite spectral triple is given and the section is concluded with the definition of a fuzzy space.

The final section of the chapter introduces the fuzzy spaces examined in this thesis. The philosophy of why making the algebra of functions noncommutative preserves the symmetry of the fuzzy sphere is given. The metric on the commutative 2-sphere is given as the induced metric from the embedding into $\mathbb{R}^3$. It is shown to be equivalent to the canonical intrinsic form. The Dirac operator for the 2-sphere is then also given in terms of the embedding of the sphere in $\mathbb{R}^3$. As the correspondence with the fuzzy sphere Dirac operator is clearly seen in this form. The spectral triple for the fuzzy sphere is then described. Introducing both the Grosse-Prešnajder Dirac operator [51] and the Dirac operator given by Barrett in [46]. A discussion of the other constructions of the fuzzy sphere is given.

The flat 2-torus is next introduced. A discussion of the modular transformations that send the torus to a parallelogram and how they affect the metric. How they also affect the Dirac operator is discussed with emphasis on the changes to the spectrum. The Dirac operator for the fuzzy torus is then presented, highlighting some of the subtleties that arise in the noncommutative geometry setting. Such as the spin structure of a fuzzy torus being fixed based upon its shape.

The procedure of how the random geometries are created via a random matrix model is then described. The choice of action that is investigated is given and the unsuitability of the spectral action for random matrix models is discussed. The location of the phase transitions for the types of fuzzy spaces studied in [63] are also provided.

Chapter 3 is mostly based on the joint work by the author, John W. Barrett and Lisa Glaser, which resulted in the following publication [64]. It introduces the spectral measures investigated in this thesis and showcases them being applied to the fuzzy

spaces introduced in chapter 2. Weyl's law for the Dirac operator is stated and then adapted to fuzzy geometry to try to measure the dimension and volume. It's limitations are discussed and used as justification for the introduction of the more advanced methods of heat kernel asymptotics. How the volume and dimension are encoded in the heat kernel is described, and how to use the spectral zeta function to isolate each quantity is outlined. This also adapted and then applied to the fuzzy spaces.

The spectral dimension of quantum gravity studies [12] is introduced and a discussion about its limitations when applied to fuzzy spaces is given. A modification called the *spectral variance* is introduced as a remedy for problems the spectral dimension faces. Both quantities are applied to the fuzzy sphere and fuzzy tori, to justify their use as dimensional measure. The random fuzzy spaces are then examined with this measure.

The question of the volume of a fuzzy space is then addressed. Two methods are investigated, both of which are based upon singular traces of operators. The first method is based upon the Dixmier trace, which is often used in noncommutative geometry to define the noncommutative integral [65]. It is shown to be related to the spectral zeta function, which is used to extract the volume from the heat kernel asymptotics. The second method is based upon the work by Stern [66]. It is similar to the Dixmier trace, in that it expresses the residues of the spectral zeta function as a limit of partial series. It also utilises the same heat kernel asymptotic expansions as the Dixmier trace. The main advantage of the Stern volume is that it converges much faster than the Dixmier trace. The Stern procedure also provides a methodology to access the subsequent poles of the spectral zeta function. Which historically required analytic continuation and a spectral expression did not exist. These measures are applied to the fuzzy sphere and tori to justify their use on the random geometries.

The actions investigated in the Monte Carlo matrix model allow for an arbitrary scaling of the Dirac operators. This scaling of the random geometries becomes im-

portant when investigating their volumes. A method to counteract it is introduced and justified by examining the volume measures.

The final spectral measure investigated is again based upon the spectral zeta function. It is a spectral distance measure between two geometries as introduced in [67]. With the zeta distance equalling zero if and only if the spectra are the same. This tool allows for the quantitative comparison of fuzzy spaces to continuum spaces, which was done for the fuzzy sphere and fuzzy torus. This analysis showed that as the matrix size is increased, both the fuzzy sphere and fuzzy torus get closer to their continuum counterparts, and subsequent fuzzy spaces get closer to each other. This zeta distance is then used to measure how similar the random geometries to each other and to the fuzzy sphere. This was investigated at various values of action parameter. Providing more evidence that the fuzzy spaces are closer to continuum geometries at the phase transition.

Chapter 4 is concerned with investigating the role of symmetry in the construction of Dirac operators for fuzzy spaces. This chapter is somewhat exploratory, and outlines the progress that has been made in understanding the role symmetry could have in constructing new fuzzy spaces. Using the sphere as a guiding example, the necessary differential geometry to describe the Dirac operator on a manifold which possess a Lie group symmetry is given. The sphere is an example of a coadjoint orbit, which are naturally symplectic manifolds and have a canonical Kähler structure. They can also be described as homogeneous spaces and additionally have a Lie algebraic description via representation theory. Such an abundance of different descriptions makes them perfect for investigating using the tools of noncommutative geometry. However, not every coadjoint orbit is a spin manifold, which is required for the duality that the construction of fuzzy spaces is based upon. The necessary condition for a coadjoint orbit to be a spin manifold is given as the existence of a lift of the isotropy representation to a spin group. As the Dirac operator is of primary interest in this thesis, a description of how to encode the Lie group symmetry within the Dirac operator is outlined. With the aim of applying an analogous procedure in the fuzzy geometry setting. It is shown that the implementation of an $SU(2)$ symme-

try on the type $(1,3)$ and $(0,3)$ fuzzy spaces reduces the available Dirac operators, to those that contain the fuzzy spheres as described by the Grosse-Prešnadjer and Barrett operators, but also allowing others. The metric data of all these operators is the same and equal to that of the fuzzy spheres. This is interpreted as the connection not being the Levi-Civita connection for the round metric. Which could mean that fuzzy spaces, as currently described, have torsion.

The final chapter is a summary of the results of the thesis and their implications. It also contains a discussion of the possible paths of future research that are a consequence of the work in this thesis.

2

BASIC NONCOMMUTATIVE GEOMETRY AND FINITE SPECTRAL TRIPLES

The objects studied in this thesis are noncommutative versions of spectral triples in t he duality of Connes [35]. The spectral triples in this duality require the presence of a spin structure and a Dirac operator. The presence of a spin structure is topologically restricting. Meaning that this duality is not for all Riemannian manifolds, but only those that satisfy a certain cohomological condition - that the second Stiefel-Whitney class vanishes, see [62] for the details.

Despite this seeming restriction, spin manifolds are of integral importance in theoretical physics. Given that spinors, which are sections of a spinor bundles over a spin manifold, are how the fundamental fermions of physics are formally described. The Dirac operator is a vital object of interest when studying fermions, as it encodes the dynamics of fermions via the Dirac equation. It is thus a necessary component of the Lagrangian for the standard model of particle physics. It is also used in the study of quantum field theory in curved spacetime as it encodes the geometry of the space via the veilbeins it contains (the details of this will be seen shortly).

So exploring extensions of this duality into the noncommutative world is a reasonable endeavour. Pseudo-Riemannian manifolds are not considered in this thesis, but are vital for physics. As the models of physics are typically conducted on a Lorentzian manifold. All of the spin geometry discussed here extends to the pseudo-Riemannian case.

2.1 A PRIMER ON SPIN GEOMETRY

Spin geometry was born out of the mathematics put forward by Dirac to describe the relativistic electron [68]. It has grown to become a powerful topic and has far reaching applications in both mathematics and physics. The modern mathematical treatment of spin geometry is documented comprehensively in the text by Lawson and Michelsohn [62]. This text is mainly concerned with the Atiyah-Singer index theorem which is a powerful statement about elliptic differential operators. However the details surrounding this theorem are not immediately pertinent to the topics discussed in this thesis. However much of the background on Clifford algebras and Clifford modules in this thesis can be sources to this text. The book by Friedrich [69] and the book by Berline et all [70] are exemplary introductions to the role of spin geometry and Dirac operators in the study of Riemannian geometry. Many results used in this thesis can be found in these texts.

A constructive approach is outlined here, where some of the finer details such as existence are omitted. Starting with the definition of a Clifford algebra bundle. Given a Riemannian manifold, M, a Clifford algebra[1], $\mathrm{Cl}(T_xM, Q_g)$, can be defined at each point of the manifold. Where T_xM is the tangent vector space at the point x and Q_g is a quadratic form induced by the metric g defined as follows:

$$Q_g(X_x) = g_x(X_x, X_x) \quad \forall X_x \in T_xM \tag{4}$$

The Clifford algebra $\mathrm{Cl}(T_xM, Q_g)$ is then the algebra generated by all $X_x \in T_xM$ and the multiplicative unit $1 \in \mathbb{R}$ such that the multiplication obeys $X_x \cdot X_x = -Q(X_x)1$. The Clifford algebra bundle is constructed by letting x vary over M.

Definition 1. The Clifford algebra bundle $\mathrm{Cl}^+(TM)$ is the fibre bundle over M, where the fibres are the Clifford algebras $\mathrm{Cl}(T_xM, Q_g)$ and the transition functions

1 A more detailed examination of Clifford algebras is given later in section 2.2.3

are inherited from the tangent bundle TM, $t_{ij}\colon U_i \cap U_j \to SO(n)$, where $n = dim(M)$ and their action on $\mathrm{Cl}(T_xM, Q_g)$ is given by:

$$t_{ij}(v_1 v_2 \dots v_k) = t_{ij}(v_1) \dots t_{ij}(v_k) \tag{5}$$

Given a Clifford algebra bundle, $\mathrm{Cl}^+(TM)$ the algebra of continuous real-valued sections can be defined $\mathrm{Cliff}^+(M) := \Gamma(\mathrm{Cl}^+(TM))$. By replacing Q_g with $-Q_g$ and an analogous but different in general Clifford algebra bundle and its sections can be defined, $\mathrm{Cl}^-(TM)$ and $\mathrm{Cliff}^-(M)$. By complexifying the Clifford algebras $\mathbb{C}\mathrm{l}(T_xM) = \mathrm{Cl}^+(T_xM) \otimes \mathbb{C}$, the complex Clifford algebra bundle, $\mathbb{C}\mathrm{l}(TM)$, can be defined. With the space of continuous sections defined by:

$$\mathbb{C}\mathrm{liff}(M) := \mathrm{Cliff}^+(M) \otimes_{\mathbb{R}} \mathbb{C}. \tag{6}$$

To pass from a Clifford algebra bundle to a spin bundle, some additional requirements. These are topological in nature, but are expressed here as the existence of a bundle isomorphism and the existence anti-linear operator.

Definition 2. A Riemannian manifold, M, is said to be *spin^c* if there exists a vector bundle $\pi : S \to M$ such that there is an algebra bundle isomorphism:

$$\mathbb{C}\mathrm{l}(TM) \simeq End(S) \qquad M \text{ is even dimensional} \tag{7}$$

$$\mathbb{C}\mathrm{l}^0(TM) \simeq End(S) \qquad M \text{ is odd dimensional.} \tag{8}$$

Where $\mathbb{C}l^0(TM)$ is the *even* part of the Clifford algebra (see section 4.1.4), and $End(S)$ is the endomorphism bundle of S, i.e. the fibre bundle with fibres $End(S_x)$ where $S_x = \pi^{-1}(x)$ for $x \in M$,. In such a case[2], the pair (M, S) is said to be a *spin^c structure* for M.

2 This definition looks a little contrived without knowing the motivation for it, which arises when considering Euclidean space $\mathbb{R}^n$ with the standard metric, δ. It can be shown that $\mathbb{C}\mathrm{l}(\mathbb{R}^{2n}) \simeq M_n(\mathbb{C})$ and $\mathbb{C}\mathrm{l}^0(\mathbb{R}^{2n+1}) \simeq M_n(\mathbb{C})$. And this definition is the generalisation to arbitrary Riemannian manifolds.

If a spin^c structure (M, S) exists, then the bundle $\pi : S \to M$ is to referred to as the *spinor bundle* and the sections as *spinors*. Let $\Gamma(S)$ denote its set of smooth sections. To form a Hilbert space of square integrable spinors that is used in physical settings, the space of sections needs to be completed in the following way.

Definition 3. The *Hilbert space of square-integrable spinors*, denoted $L^2(S)$, is defined by the completion of $\Gamma(S)$ under the norm induced by the following inner product for $\phi_i \in \Gamma(S)$:

$$(\phi_1, \phi_2)_M = \int_M \langle \phi_1, \phi_2 \rangle_S \sqrt{g} \mathrm{d}x. \tag{9}$$

The inner product of spinors is just the Clifford inner product at each point of the manifold. Where the inner product on the Clifford algebras can be defined on basis elements, from its quadratic form as in the following definition.

Definition 4. Let $\{e_i\}$ be an orthonormal basis of V. Setting $\alpha = e_1 \dots e_p$ let $\hat{\alpha} = e_p \dots e_1$, then the inner product is

$$\langle e_{i_1} e_{i_2} \dots e_{i_p}, e_{j_1} e_{j_2} \dots e_{j_q} \rangle := \langle 1, e_{i_p} \dots e_{i_2} e_{i_1} e_{j_1} e_{j_2} \dots e_{j_q} \rangle \tag{10}$$

$$= \begin{cases} 0 & \text{if } p \neq q \\ 0 & \text{if } e_{i_k} \neq e_{j_k} \text{ for any } k \\ Q(e_{i_1}) \dots Q(e_{i_p}) & \text{otherwise} \end{cases} \tag{11}$$

and can be extended linearly to arbitrary elements of $Cl(V, Q)$ by $\langle \alpha, \beta \rangle = \langle 1, \hat{\alpha}\beta \rangle$

However, the presence of a spin^c structure is not enough for a model of reality, as each spinor has to be paired with its anti-particle spinor. To be able to do this mathematically requires us to have not just a spin^c manifold but a *spin manifold*.

Definition 5. A Riemannian spin^c manifold is called a *Riemannian spin manifold* if there exists an anti-unitary operator $J_M \colon \Gamma(S) \to \Gamma(S)$ such that:

1. J_M commutes with the action of real valued continuous functions on $\Gamma(S)$.

2. J_M commutes with $\mathrm{Cliff}^-(M)$ in the even cases and with $\mathrm{Cliff}^-(M)^0$ in the odd cases.

The pair (S, J_M) is referred to as a *spin structure* on M and J_M is referred to as the *charge conjugation* on M.

Another important structure which is necessary for understanding the basic objects in noncommutative geometry is the notion of a *chirality operator*. Let $\{\gamma_a\}_{a=1}^n$ be the generators of $\mathrm{Cliff}^+(M)\big|_U$ for some some open neighbourhood, U, of M and let $\{x_a\}_{a=1}^n$ be local coordinates on U. The γ_a satisfy: $\gamma_a\gamma_b + \gamma_b\gamma_a = 2g(\partial_a, \partial_b)$ and if an orthonormal basis is chosen for $\Gamma(TM)\big|_U$ then the γ_a satisfy the relation:

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}. \tag{12}$$

Definition 6. The *chirality operator* can be constructed from the γ_a as follows:

$$\gamma_M = (-i)^m \gamma_1 \dots \gamma_n \tag{13}$$

where $m = n/2$ (when n is even) or $m = (n-1)/2$ (when n is odd).

The final object needed is that of a Dirac operator.

2.1.1 *The Dirac Operator*

The Dirac operator can be thought of a "square root" of the Laplacian of a space and contains a lot of information. The Dirac operator describes the dynamics of spinors via the Dirac equation and it also encodes the metric information as is shown later.

To define the Dirac operator, the notion of a spin connection is required. Given a Riemannian spin manifold M with spin structure (S, J_M). Let $\{E_a\}$ be a local orthonormal basis for local patch $TM\big|_U$ then $g(E_a, E_b) = \delta_{ab}$ when restricted to this

patch. Let θ^a be the duals to E_a, then the Levi-Civita connection in this basis acts on vectors and one forms in the following way:

$$\nabla E_a = \widetilde{\Gamma}_a{}^b{}_c \, \mathrm{d}x^c \otimes E_b \tag{14}$$

$$\nabla \theta^a = -\widetilde{\Gamma}^a{}_{bc} \, \mathrm{d}x^b \otimes \theta^c \tag{15}$$

And recall that if E_a is an orthonormal basis orthonormal basis for $TM|_U$ then the gamma matrices satisfy: $\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}$.

Definition 7. The spin connection ∇^S on the spinor bundle $(S, M, End(S), f)$ is the lift of the Levi-Civita connection to the spinor bundle and is locally is given by:

$$\nabla^S_a \psi(x) = \left(\partial_a - \frac{1}{4} \widetilde{\Gamma}^b{}_{ac} \gamma^c \gamma_b \right) \psi(x). \tag{16}$$

The notion of Clifford multiplication is also necessary to define the Dirac operator.

Definition 8. *Clifford multiplication* is defined as the linear map:

$$c \colon \Omega^1(M) \times \Gamma(S) \to \Gamma(S) \tag{17}$$

$$(\omega, \psi) \to \omega^\# \cdot \psi \tag{18}$$

where $\Omega^1(M)$ is the space of 1-forms on M and $\omega^\#$ vector field in $\Gamma(TM)$ associated to ω. The vector field acts an endomorphism on the fibres of S via the embedding $\Gamma(TM) \to \mathrm{Cliff}^+(M) \subset \Gamma(\mathrm{End}(S))$. Choosing local coordinates for $U \subset M$ the one forms can be written as $\omega|_U = \omega_a \, \mathrm{d}x^a$ and the Clifford multiplication as follows

$$c(\omega, \psi)|_U = \omega_a (\gamma^a \psi)|_U. \tag{19}$$

Where the $\mathrm{d}x^a$ have been identified with ∂_a via the metric and then embedded in $\mathrm{Cliff}^+(M)$ and become the γ^a.

The pair $(\Gamma(S), c)$ is a called *Clifford module* and will play an important roll in the noncommutative setting. For completeness the definition for a Clifford module has been included below:

Definition 9. A *Clifford module* over a compact Riemannian manifold (M, g) is a pair $(\Gamma(E), c)$ where $\Gamma(E)$ is the sections of a smooth vector bundle, E, and c is a $\mathrm{Cl}(TM)$-module homomorphism from $\Gamma(\mathrm{Cl}(TM)) \to \Gamma(\mathrm{End}(\Gamma(E)))$.

The Dirac operator for a Riemannian manifold is now ready to be defined. Its expression in local coordinates has been given to indicate why so much preparatory work was necessary.

Definition 10. The *Dirac operator* for a spin manifold M with spin structure (S, J_M) is the composition of spin connection with Clifford multiplication and can be expressed in local coordinates as:

$$\not{D}_M = c \circ \nabla^S \colon \Gamma(S) \to \Gamma(S) \tag{20}$$

$$\not{D}_M \psi(x) = -i\gamma^a \left(\partial_a - \frac{1}{4} \widetilde{\Gamma}^b{}_{ac} \gamma^c \gamma_b \right) \psi(x) \tag{21}$$

To see how the Dirac operator encodes the metric, it is useful to look at the definition of a Dirac operator in terms of vielbeins. Vielbeins are defined in the following way. Take an orthonormal basis $\{e_a\}$ for $\Gamma(TM)$ such that[3] $g(e_a, e_b)(x) = \delta_{ab}$. This can expressed in terms of the coordinate basis $\{\partial_\mu\}$ as follows $e_a(x) = e^\mu{}_a(x)\partial_\mu(x)$. A vielbein is defined to be $e^\mu{}_a$; the invertible transformation matrix, however the name often extends to the orthonormal basis also. Also note that the metric equation about can now be rewritten as $g_{\mu\nu}(x) e^\mu{}_a(x) e^\nu{}_b(x) = \delta_{ab}$ or equivalently

$$g_{\mu\nu}(x) = e_\mu{}^a(x) e_\nu{}^b(x) \delta_{ab}, \tag{22}$$

3 This is for Riemannian metrics, for pseudo-Riemannian metrics the δ in the relation is replaced with the corresponding pseudo-Riemannian equivalent.

where $e^{\mu}{}_{a}(x)e^{a}{}_{\nu}(x) = \delta^{\mu}{}_{\nu}$ and $e^{\mu}{}_{a}(x)e^{b}{}_{\mu}(x) = \delta_{a}{}^{b}$. The Dirac operator in terms of vielbeins is then:

$$\not{D}_M = -i\gamma^a e_a{}^b \nabla^S_b. \tag{23}$$

So given a Dirac operator, the vielbeins can extracted[4] and therefore the full metric can be reconstructed.

2.2 SPECTRAL TRIPLES AND FUZZY SPACES

This thesis is primarily concerned with *fuzzy spaces*. These are matrix algebras that capture some features of an ordinary manifold. The archetypical example is that of the fuzzy sphere. This is a matrix algebra that is rotationally invariant. As there are no longer a set of points on which the rotation group can act, the notion of rotational invariance needs careful thought. A precise explanation of this is presented in section 2.3.1.

Usually when you restrict to finite dimensional algebras you no longer are able to have continuous symmetries. Think about approximating a circle by finitely many points, you can no longer rotate this space by an infinitesimal amount and remain in the same space (see Figure 1). For fuzzy spaces, it is possible to have a finite dimensional algebra and retain these symmetries at the cost of commutativity.

The term "fuzzy space" is used to describe any construction of matrix algebras that emulates a normal topological space. In this thesis a fuzzy space will mean a noncommutative finite spectral triple. These are described below in full, but the main difference between this construction of fuzzy spaces and others, is the presence of a noncommutative analogue of a spinor bundle over the space. This is captured by the presence of an operator that is the analogue of the Dirac operator. As the Dirac operator specifies a metric tensor, spectral triples are key research entities into studying noncommutative Riemannian geometry. With current efforts to loosen the restriction from spin manifolds to arbitrary Riemannian manifolds. As the existence

4 This is explicitly demonstrated for the case of the sphere in section 2.3.1.

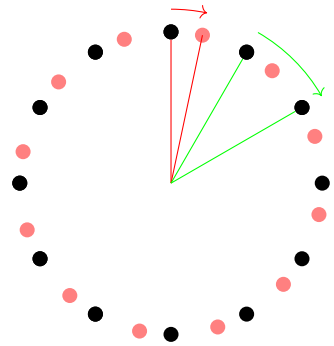


Figure 1.: An example of a circle approximated by finitely many points. The red lines indicates a rotation that does not preserve the set of points. The green line indicates a rotation that does preserve the set of points. So the approximated circle does not possess the full rotation group, $SO(2)$, symmetry.

of a spin structure is quite restricting in a topological sense. There is also work to try and include pseudo-Riemannian geometries into the noncommutative geometry framework.

2.2.1 *Spectral Triples*

The details for a finite spectral triple as presented by Barrett in [46] are given in full below. The details of a infinite spectral triple are omitted as it's not pertinent to the current work and can be found in [71].

Definition 11. A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$, where:

- $\mathcal{A}$ is a $*$-unital algebra represented as bounded operators on $\mathcal{H}$.
- $\mathcal{H}$ is a Hilbert space
- D is a self-adjoint operator on $\mathcal{H}$ such that the resolvent $(i + D)^{-1}$ is a compact operator and $[D, a]$ is bounded for each $a \in \mathcal{A}$.

Table 1.: The K0 dimension, n, of a real spectral triple is determined by the signs $\epsilon, \epsilon', \epsilon''$.

	K0 Dimension							
n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1	1	-1	1	1	1	-1	1

Additional structures are required of noncommutative geometries if they are to be appropriate generalisations of Riemannian spin geometries. The first is a $\mathbb{Z}_2$-grading γ on the Hilbert space $\mathcal{H}$ such that:

$$\gamma a = a\gamma \quad (\forall a \in \mathcal{A}), \qquad \gamma D = -D\gamma \tag{24}$$

If such a γ exists then the spectral triple is said to be *even*. The second is an anti-linear map $J\colon \mathcal{H} \to \mathcal{H}$ such that:

$$J^2 = \epsilon, \qquad JD = \epsilon' DJ, \qquad J\gamma = \epsilon'' \gamma J \quad (\text{when } \gamma \text{ exists}) \tag{25}$$

where $\epsilon, \epsilon', \epsilon''$ take values from table 1. If such a J exists then the spectral triple is said to be *real* and J is referred to the as the *real structure*.

The values for $\epsilon, \epsilon', \epsilon''$ are determined by the *KO dimension* of the real spectral triple. Which for the spectral triple of a Riemannian spin manifold will be equal to the dimension of the manifold modulo 8. The specific values of $\epsilon, \epsilon', \epsilon''$ for each K0 dimension are given in table 1.

Two extra conditions are imposed, which are chosen to keep the framework from straying too far away from commutative geometry. Firstly, using the J operator the *opposite algebra* can be defined, which describes the right action of the algebra. If $a \in \mathcal{A}$ then $a^\circ = Ja^*J^{-1}$ defines a right action. Thus the right action of the algebra

on the Hilbert space is $\psi b = b^\circ \psi$. The first requirement is that the left and right actions of the algebra commute:

$$[a, b^\circ] = 0 \qquad \textit{Order zero condition}. \tag{26}$$

This is used to model the fact that for functions $f \in C^\infty(M)$ and ψ spinors of some Riemannian spin manifold commute. They satisfy $(f\psi)(x) = f(x)\psi(x) = (\psi f)(x)$. In other words, in commutative geometry the left and right actions of the algebra of functions are indistinguishable. It is this commutative geometry feature that is preserved even when the 'functions' no longer commute with each other.

The second condition is to do with the fact that commutation with the Dirac operator should also commute with the right action:

$$[[D, a], b^\circ] = 0 \qquad \textit{First order condition}. \tag{27}$$

Looking at commutative geometry again, the Dirac operator can be expressed as in eq. (23), and thus:

$$[D, f]\,\psi = -i\gamma^a e_a^b (\nabla_b^S f)\psi. \tag{28}$$

Which acts as multiplication by a function and so will commute with a right multiplication by a function. It is this property that is required to persist in our noncommutative geometry. Also note, that there are noncommutative geometry models where higher order conditions are available, notably the work by Boyle and Farnsworth [72].

The following spectral triple for a Riemannian spin manifold satisfies the two conditions. It is this analogues of this object that are extended to be noncommutative in what follows.

Definition 12. The real spectral triple for a Riemannian spin manifold, M, with spin structure (S, J_M) is the structure:

$$\left(C^\infty(M), L^2(S), \not{D}_M; \gamma_M, J_M\right), \tag{29}$$

where $L^2(S)$ is the Hilbert space as the space of square integrable spinors and $C^\infty(M)$ is the algebra of smooth functions. And where $\not{D}_M$, γ_M and J_M are the Dirac operator, chirality operator and charge conjugation as defined in the previous section.

2.2.2 *Axioms for a finite real spectral triple*

A few basic definitions are required before the axioms can be stated.

Definition 13. An *algebra*, $\mathcal{A}$, is a vector space over a field $\mathbb{F}$ (usually taken to be $\mathbb{R}$ or $\mathbb{C}$), with an associative bilinear binary operation. Let $a, b \in \mathcal{A}$, let the binary operation be denoted by $(a, b) \mapsto ab \in \mathcal{A}$ and associativity requires that $(ab)c = a(bc)$. A *unital algebra* is an algebra with an element $1 \in \mathcal{A}$ such that $1a = a1 = a$ for all $a \in \mathcal{A}$. An algebra is said to be *commutative* if $ab = ba$ for all $a, b \in \mathcal{A}$ and is said to be *noncommutative* otherwise. The dimension of an algebra is the dimension of it as a vector space.

A key example of a unital noncommutative algebra is the space of n dimensional matrices over the complex numbers, denoted $M_n(\mathbb{C})$ (note that the matrices over $\mathbb{R}$ are also an example).

Definition 14. An algebra over $\mathbb{C}$ is said to be an *involutive algebra* or a $*$*-algebra* if it possesses a linear map $* : \mathcal{A} \to \mathcal{A}$ such that for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$:

1. $(a^*)^* = a$ for all $a \in \mathcal{A}$
2. $(ab)^* = b^* a^*$
3. $(\alpha a)^* = \bar{\alpha} a^*$,

where $\bar{\alpha}$ is complex conjugation.

The n-dimensional complex matrices form a unital $*$-algebra with the involution given by the conjugate transpose. A comparison between spectral measures of the averaged spectrum and the ensemble average of spectral measures is considered below. A priori these methods have no need to produce similar results.

Definition 15. Let $\mathcal{A}$ be a finite dimensional $*$-algebra

1. A pair $(\pi, \mathcal{H})$ is called a $*$-algebra representation of $\mathcal{A}$ if $\pi : \mathcal{A} \to L(\mathcal{H})$ is a $*$-algebra homomorphism into the space of linear operators on $\mathcal{H}$, denoted by $L(\mathcal{H})$.

2. A subspace of $V \subset \mathcal{H}$ is said to be an invariant subspace of $L \in L(\mathcal{H})$ if $Lv \in V$ for all $v \in V$.

3. A representation is said to be irreducible if the only $\pi(A)$-invariant subspaces of $\mathcal{H}$ are $\mathcal{H}$ and $\{0\}$, otherwise the representation is said to be reducible.

4. A representation, $(\pi, \mathcal{H})$ of $\mathcal{A}$ is said to be faithful if $\pi(a) = \pi(b)$ implies $a = b$ for all $a, b \in \mathcal{A}$

Definition 16. An inner product space is a vector space, V over a field $\mathbb{F}$ with a map $\langle , \rangle : V \times V \to \mathbb{F}$ satisfying the following properties:

- Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for any $x, y \in V$
- Linearity: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for $a, b, \in \mathbb{F}$ and $x, y, z \in V$
- Positive-definite: $\langle x, x \rangle > 0$ for $x \in V \backslash \{0\}$

If the vector space is finite dimensional then is called a finite dimensional inner product space.

A useful concept is that of a structure preserving map from an algebraic object to itself.

Definition 17. An *endomorphism* of is a homomorphism from an object to itself.

Thus the precise meaning of an endomorphism changes depending on what is being examined. An endomorphism of a vector space is a linear map to itself, an endomorphism of an algebra is an algebra homomorphism to itself. If an endomorphism is also an isomorphism it is called an *automorphism*.

Definition 18. Given an inner product space $(V, \langle , \rangle)$, an operator A is said to be the adjoint of an operator B if $\langle Av, w\rangle = \langle v, Bw\rangle$, the adjoint of an operator is denoted by A^*. An operator is said to be self-adjoint if $A^* = A$.

The full algebraic definition of a finite real spectral triple is now given.

Definition 19 (Axioms for a finite real spectral triple)**.** A finite real spectral triple consists of the following data:
Objects:

1. an $*$-algebra $\mathcal{A}$ over $\mathbb{R}$
2. a finite dimensional inner product space $\mathcal{H}$, with a Hermitian inner product $\langle \cdot, \cdot \rangle$
3. a faithful action[5] $\rho : \mathcal{A} \to End(\mathcal{H})$ such that $\rho(a^*) = \rho(a)^*$
4. an operator $\Gamma : \mathcal{H} \to \mathcal{H}$ such that $\Gamma^* = \Gamma$, $\Gamma^1 = 1$ called the chirality operator.
5. an anti-unitary[6] operator $J : \mathcal{H} \to \mathcal{H}$ called the real structure.
6. an integer s that is defined modulo 8
7. a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$

Conditions

(i) $\Gamma\rho(a) = \rho(a)\Gamma$ for all $a \in \mathcal{A}$

(ii) $J^2 = \epsilon$

(iii) $[\rho(a), J\rho(b)J^{-1}] = 0$ for all $a, b \in \mathcal{A}$

(iv) $J\Gamma = \epsilon''\Gamma J$

(v) $D\Gamma = -\Gamma D$ if s is even and $D\Gamma = \Gamma D$ if s is odd.

5 A faithful action of an algebra can be called a faithful representation. It means that $\rho(a) = \rho(b) \implies a = b$ for an $a, b \in \mathcal{A}$

6 An anti-unitary operator, A, is such that $\overline{(Av, Aw)} = \overline{(v, w)}$, $A(v + w) = Av + Aw$ and $A(\alpha v) = \overline{\alpha} Av$. Note that for a Hermitian inner product $\overline{(v, w)} = (w, v)$.

(vi) $JD = \epsilon' DJ$

(vii) $[[D, \rho(a)], J\rho(b)J^{-1}] = 0$ for all $a, b \in \mathcal{A}$

where the signs $\epsilon, \epsilon', \epsilon''$ are taken from table 1.

Note that $\mathcal{A}$ is not explicitly defined to be a matrix algebra because of the following proposition.

Proposition 1 ([73]). *If $\mathcal{A}$ is a *-algebra over $\mathbb{R}$ that is faithfully represented on a finite dimensional Hilbert space by an $\mathbb{R}$-linear $*$-algebra map $\pi : \mathcal{A} \to End(H)$, then $\mathcal{A}$ is a matrix algebra of the form*

$$\mathcal{A} \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{F}_i) \tag{30}$$

where $F_i = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ depending on i.

Remark 1 ([74]). *Note that if $\mathcal{A}$ is required to be a finite dimensional C^* algebra, then the decomposition can be restricted to $\mathbb{F} = \mathbb{C}$.*

The following definition is useful when considering the random geometries later in the thesis.

Definition 20. A *fermion space* is defined to be specified by the objects $1 - 6$ and conditions $i - iv$ in definition 19, $(\mathcal{A}, \mathcal{H}, s, J, \Gamma)$.

Given a fixed fermion space, the following question can be asked, what possible operators D exist? To define a *path integral over geometries* that is based upon the Dirac operator, D, the notion of a fermion space will prove to be useful.

The conditions $v - vii$ in definition 19 are all $\mathbb{R}$-*linear*, meaning that the space of Dirac operators on a fermion space form a $\mathbb{R}$-vector space. Let this vector space be denoted by $\mathcal{G}$.

Remark 2. *Note that this doesn't not account for unitarily equivalent Dirac operators. The space formed by taking the quotient of the unitarily equivalent Dirac operators is a moduli space and was studied in [75].*

Definition 21. Let R be a ring and 1_R be the multiplication identity. A *left R-module*, $\mathcal{M}$, consists of an abelian group $(\mathcal{M}, +)$, and an operation $\cdot : R \times \mathcal{M} \to \mathcal{M}$ such that $\forall r, s \in R$ and $\forall x, y \in \mathcal{M}$:

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(rs) \cdot x = r \cdot (s \cdot x)$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $1_R \cdot x = x$

The difference between an algebra and a module is that an algebra is a module over a ring, where the ring is also a field. The notion of a *right R-module* also exists, with nearly identical axioms. A *R-bimodule* is simultaneously a left and right R module.

Remark 3. *Note that Ja^*J^{-1} defines a right action of the algebra $\mathcal{A}$ on $\mathcal{H}$. Let $\psi \triangleleft b := J\rho(b)^*J^{-1}\psi$, then $(\psi \triangleleft b) \triangleleft c = J\rho(c)^*J^{-1}J\rho(b)^*J^{-1}\psi = J\rho(c)^*\rho(b)^*J^{-1}\psi = J\rho(bc)^*J^{-1}\psi = \psi \triangleleft (bc)$. This promotes $\mathcal{H}$ to a $\mathcal{A}$*-bimodule.

Definition 22. An irreducible R-module is an R-module, M, such that the only submodules (subspaces that are R-modules in their own right) are M and $\{0\}$.

An irreducible $\mathcal{A}$-bimodule, $\mathcal{H}$, is a vector space that carries a left and a right action of $\mathcal{A}$, that is both left and right irreducible with respects to those actions. More specifically there is no subspace $W \subset \mathcal{H}$ such that $\pi(a)W \subset W$ and no space $X \subset \mathcal{H}$ such that $X\lambda(a) \subset X$. The left and right actions are either irreducible or they can be decomposed into a direct sum of irreducible representations. Denoting $\pi = \pi_1 \oplus \dots \pi_n$ as the left action with it's decomposition and $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_m$ as the right action with it's decomposition. The following expression of $\mathcal{H}$ is possible:

$$\mathcal{H} = \oplus_{ij} H_{ij}, \tag{31}$$

where H_{ij} is an irreducible left-π_i and right-λ_j module.

A simple but important example of a real finite spectral triple is:

Definition 23. A type $(0, 0)$ matrix geometry is a finite real spectral triple of K0 dimension $s = 0$ and the following objects: $(\mathcal{A}_0, \mathcal{H}_0, D_0 = 0; \Gamma_0 = 1, J_0)$ such that

the each irreducible $\mathcal{A}_0$-bimodule in the decomposition of $\mathcal{H}_0$ are inequivalent (not unitarily equivalent).

It can seen, using the axiom (iv) for even s, that $D = 0$ is the only option for type $(0,0)$ matrix geometries, as the requirement D anti-commutes becomes $D1 = -1D$.

The statement in the above definition about each irreducible $\mathcal{A}_0$-bimodule being inequivalent is expressed in that none of the H_{ij} in the direct sum decomposition of eq. (31) are repeated.

Example 1. *Take $\mathcal{H}$ to be a $\mathbb{C}$-linear vector subspace of $M_n(\mathbb{C})$ with a faithful representation $\rho : \mathcal{A} \to End(\mathcal{H}) \simeq M_n(\mathbb{C})$ given by matrix multiplication, so that $\rho(a)m \in \mathcal{H}$ and $m^* \in \mathcal{H}$ for all $m \in \mathcal{H}$ and for all $a \in \mathcal{A}$. The inner product on $\mathcal{H}$ is given by $(m, m') = Tr(m^* m')$, the real structure is given by $J(m) = m^*$. Thus the right action of $\mathcal{A}$ is $J\rho(a)^* J^{-1} m = J\rho(a)^* m^* = m\rho(a)$ which is just right matrix multiplication. J is anti-unitary: $(Jm, Jm') = Tr((Jm)^* Jm') = Tr((m^*)^*(m')^*) = Tr(m(m')^*) = (m', m) = \overline{(m, m')}$. The action of a^* is given by $(m, \rho(a^*)m') = Tr(m^* \rho(a^*)m') = Tr((\rho(a^*)^* m)^* m') = (\rho(a^*)^* m, m')$, let $\rho(a^*) = \rho(a)^*$ so a^* is the adjoint of a in $\mathcal{H}$.*

The following result tells us that example 1 is the one example of type $(0,0)$ geometry one needs to study.

Proposition 2 ([46])**.** *A type $(0,0)$ matrix geometry is isomorphic to one constructed in Example 1.*

Another way of generating fermion spaces is by taking Clifford modules. The construction of a general so-called (p,q) module is presented next, as it will be used extensively later on.

2.2.3 *Clifford Algebras*

The construction of fuzzy spaces heavily relies on Clifford algebras, so a detailed account of the objects needed is presented here.

Definition 24. Given a vector space V (over $\mathbb{F} = \mathbb{R}, \mathbb{C}$) and a quadratic form, Q on V. The *Clifford Algebra* $\mathrm{Cl}(V, Q)$ is defined as the algebra generated (over $\mathbb{F}$) by the vectors $v \in V$ and the multiplicative unit 1_F such that:

$$v \cdot v = v^2 = -Q(v)1_F \tag{32}$$

In this thesis, Clifford algebras are defined over $\mathbb{R}$ or $\mathbb{C}$ however, much of what follows is valid when the field has characteristic not equal to two. A Clifford algebra can also be defined by taking a bilinear form, B, instead of a quadratic form and defining the quadratic form by: $Q(v) = B(v, v)$. One can also construct a bilinear form from a quadratic form via *polarisation*: $B(u, v) = \frac{1}{2}\left(Q(u + v) - Q(u) - Q(v)\right)$. Using this polarisation it can be shown for $u, v \in V$ that:

$$uv + vu = -2B(u, v) \tag{33}$$

If $\{e_i\}$ is an orthonormal basis for the vector space V with respects to the bilinear form B then they satisfy:

$$e_i e_j + e_j e_i = -2B_{ij} \tag{34}$$

Let η be a non-degenerate diagonal matrix of ± 1s, with p occurrences of $+1$ and q occurrences of -1. The matrix is said to be of type (p, q) and the total dimension is given by $n = p + q$. A real Clifford algebra is then defined over $V = \mathbb{R}^n$ by:

$$e_i e_j + e_j e_i = 2\eta_{ij} \tag{35}$$

For a given (p, q) matrix η let $Cl(p, q)$ denote the Clifford algebra over $\mathbb{R}^n$.

The low dimensional Clifford algebras are presented here. From these examples the higher dimensional Clifford algebras can be constructed [62].

Example 2. *(i) Type $(0, 0)$: There are have no generators, so there is only the identity element to generate. So the space is isomorphic to $\mathbb{R}$*

(ii) Type $(0,1)$: *There is one generator, e that satisfies $e^2 = -1$, so a general element is of the form $a + be$, which is isomorphic to $\mathbb{C}$ under the map $a + be \mapsto a + bi$.*

(iii) Type $(0,2)$: *There are have two generators e_1, e_2 that both square to -1 which anti-commute, $e_1 e_2 = -e_2 e_1$. The product is not linearly decomposable in terms of $1, e_1, e_2$, in fact if we define $e_{ij} = \frac{1}{2} e_i e_j - e_j e_i = \frac{1}{2}[e_i, e_j]$, then we see that $e_i e_j = e_{ij} + \eta_{ij}$. So an arbitrary element consists of $a + be_1 + ce_2 + de_{12}$. This space is isomorphic to the quartenions $\mathbb{H}$, sending $(e_1, e_2, e_1 e_2) \mapsto (i, j, k)$. In the complex matrix representation of $\mathbb{H}$ we have that $(e_1, e_2, e_1 e_2) \mapsto (\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix})$.*

(iv) Type $(1,0)$: *This is generated by e, such that $e^2 = 1$. So we have that an element has the form $a + be$, which has product*

$$(a + be) \cdot (a + be) = 2\eta(a + be, a + be) \tag{36}$$

$$= 2\eta(a, a) + 2\eta(a, be) + 2\eta(be, a) + 2\eta(be, be) \tag{37}$$

$$= 2\eta(a, a) + 2\eta(be, be) = (a \cdot a) + (be \cdot be) \tag{38}$$

So we have that $Cl(1,0)$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

(v) Type $(2,0)$ *We have e_1, e_2 such that $e_i^2 = 1$ and $e_1 e_2 = -e_2 e_1$. So an arbitrary element is of the form $a + be_1 + ce_2 + de_1 e_2$. Given the following algebra-isomorphism*

$$(e_1, e_2, e_1 e_2) \mapsto (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \tag{39}$$

we have that $Cl(2,0) \cong M_2(\mathbb{R})$

(vi) Type $(1,1)$ *We have* e_1, e_2 *such that* $e_1^2 = 1$ *and* $e_2^2 = -1$ *and again* $e_1e_2 = -e_2e_1$ *so the general form of an element again is* $a + be_1 + ce_2 + de_1e_2$ *and under the following algebra-isomorphism*

$$(e_1, e_2, e_1e_2) \mapsto \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \tag{40}$$

So we have that $Cl(1,1) \cong M_2(\mathbb{R})$ *just like* $Cl(2,0)$.

It turns out that the following classification for Clifford algebras exists:

Theorem 1. *The Clifford algebra* $Cl(p,q)$ *is of the form* $M_{2^m}(\mathbb{F})$ *or* $M_{2^m}(\mathbb{F}) \oplus M_{2^m}(\mathbb{F})$ *where* $\mathbb{F} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$ *and* m *is such that* $n = p + q = 2m$ *or* $n = p + q = 2m + 1$. *Specifically the classification follows Table 2 where* $s = p - q \bmod 8$.

s	$Cl(p,q)$	s	$Cl(p,q)$
0	$M_{2^{n/2}}(\mathbb{R})$	4	$M_{2^{(n-2)/2}}(\mathbb{H})$
1	$M_{2^{(n-1)/2}}(\mathbb{C})$	5	$M_{2^{(n-1)/2}}(\mathbb{C})$
2	$M_{2^{(n-2)/2}}(\mathbb{H})$	6	$M_{2^{n/2}}(\mathbb{R})$
3	$M_{2^{(n-3)/2}}(\mathbb{H}) \oplus M_{2^{(n-3)/2}}(\mathbb{H})$	7	$M_{2^{(n-1)/2}}(\mathbb{R}) \oplus M_{2^{(n-1)/2}}(\mathbb{R})$

Table 2.: Classification of real Clifford algebras.

Proof. See Lawson [62, Chp. 1, 4]. □

Note that the Clifford algebras are direct sums of matrix algebras. This fact is useful when considering representations of Clifford algebras below. Another useful fact about Clifford algebras is the following.

Proposition 3. *Given a* (p,q) *Clifford algebra, the generators,* $\{e_i\}$, *and the identity generate a multiplicative finite group.*

Proof. Assuming there are n of the e_i's form the set

$$G = \{\mathbb{I}, e_i, e_{i_1}e_{i_2}, e_{i_1}e_{i_2}e_{i_3}, \ldots, e_{i_1} \cdots e_{i_n}\}. \tag{41}$$

The indices ordered so that $i_1 < i_2 < \cdots < i_n$. It will be shown that $\pm G$ forms a group with respect to multiplication. Start with $e_i e_{j_1} e_{j_2} \cdots e_{j_m}$ and examine the following cases. Either $i = j_k$ for some k, in which case the relation $e_i e_j + e_j e_i = 2\eta^{ij}$ can be used to anti-commute the e_i through until it meets it's pair, with which it then becomes either $+1$ or -1. This will yield an entry of $\pm G$ with one fewer gamma. If e_i does not appear in the element $e_{j_1} \cdots e_{j_m}$, then it can anti-commuted through until the correct ordering is achieved and it therefore lies in $\pm G$. Then any product of elements can be broken down iteratively through this process. The set is clearly finite with

$$2 \cdot \sum_{k=0}^{n} \binom{n}{k} = 2 \cdot 2^n = 2^{n+1} \tag{42}$$

elements. □

2.2.4 *Clifford Modules*

Clifford modules can now be examined, as these will be a necessary component in the definition of fuzzy spaces.

Definition 25. A (p, q)-*Clifford module* for a (p, q) Clifford algebra over the field $\mathbb{F}$ is a pair (c, V). Where V is a finite dimensional vector space over either $\mathbb{F}$ or a field containing $\mathbb{F}$, and $c : Cl(p, q) \to Hom_{\mathbb{F}}(V, V) = End_{\mathbb{F}}(V)$ is an algebra homomorphism. V is said to be a $Cl(p, q)$-module for short.

The map c is often suppressed and the notation $c(\psi)v = \psi \cdot v$ for $v \in V$ and $\psi \in Cl(p, q)$ is used when there is no confusion. This is also called *clifford multiplication*. Also note that a representation of a Clifford algebras is exactly a module for it. It is more common to refer to a real or complex *representations of a Clifford algebra* to specify what field the vector space is over.

Definition 26. A module (c, V) of a (p, q) Clifford algebra is *reducible* if the vector space V can be written as non-trivial direct sum: $W = W_1 \oplus W_2$ such that $c(\psi)(W_j) \subseteq$

W_j for every $\psi \in Cl(p,q)$. In such cases the following notation is adopted $c = c_1 \oplus c_2$ likewise. A module is called *irreducible* if it is not reducible.

Due to the finite dimensionality of the Clifford modules, the following proposition holds:

Proposition 4 ([62])**.** *Every representation of a Clifford algebra can be decomposed into a direct sum of irreducible representations.*

The following statement along with Table 2 allows one to easily select the irreducible representations for the Clifford algebras.

Theorem 2. *The natural representation of $M_n(\mathbb{F})$ as operators on $\mathbb{F}^n$ is, upto equivalence, the only irreducible representation. The algebra $M_n(\mathbb{F}) \oplus M_n(\mathbb{F})$ has two inequivalent representations given by $\rho_1(m_1, m_2) = \rho(m_1)$ and $\rho_2(m_1, m_2) = \rho(m_2)$ where ρ is the representation of $M_n(\mathbb{F})$.*

Proof. See [62, p. 32] or [76, p.653 Thm 4.3 and 4.4], realising that matrix algebras are simple, so in either case we are dealing with a semisimple algebra. □

So the module is taken to be $(c, V = \mathbb{C}^k)$ with $k = 2^{n/2}$ for n even and $k = 2^{(n-1)/2}$ for n odd. Note for odd values of n there are two inequivalent irreducible representations. Let $\mathbb{C}^k$ be equipped with the inner product $(v, w) = \overline{v} \cdot w = \sum_i \overline{v}^i w^i$. Let $c(e_i) = \gamma^i$ and $c(1_{Cl(p,q)}) = \mathbb{I}_k$ the $k \times k$ identity matrix. Note a unitary representation of a finite group can always be taken [77]. As a result, all of the γ^i are unitary. Using the standard fact that

$$(\gamma^i v, w) = \overline{(\gamma^i)_{ab} v^b} w_a = \overline{v^b (\gamma^i)_{ab}} w_a = \overline{v_b} \overline{(\gamma^i)^T_{ba}} w_a = (v, (\gamma^i)^* w), \tag{43}$$

if $(\gamma^i)^2 = 1$, then

$$(\gamma^i v, \gamma^i w) = (v, w) = (u, (\gamma^i)^2 w) = ((\gamma^i)^* v, \gamma^i w). \tag{44}$$

Consequently $((\gamma^i)^* - \gamma^i)v, \gamma^i w) = 0$ for all $v, w \in \mathbb{C}^k$. Thus $(\gamma^i)^* = \gamma^i$, i.e it is Hermitian. Similarity if $(\gamma^i)^2 = -1$, then $(\gamma^i)^* = -\gamma^i$, i.e. it is anti-Hermitian.

The chirality operator and the real structure for Clifford modules are defined, and will be used to construct the corresponding operators in the spectral triple.

Definition 27. The chirality operator for a (p,q) Clifford module is given by the ordered product

$$\widetilde{\gamma} = i^{s(s+1)/2}\gamma^1 \cdots \gamma^n \tag{45}$$

A useful fact for later is that when n is even (and so s even) $\widetilde{\gamma}$ anti-commutes with each γ^i and when n (and therefore s) is odd it commutes with each γ^i.

Definition 28. A real structure for a Clifford module is an anti-linear operator $C : V \to V$ such that $(Cv, Cw) = (w, v)$, $C^2 = \epsilon \mathbb{I}$ and $C\gamma^a = \epsilon' \gamma^a C$ for all γ^a. When the chirality operator γ exists it is required that $C\widetilde{\gamma} = \epsilon'' \widetilde{\gamma} C$, where ϵ and ϵ' are taken from 1.

Low dimensional examples of Clifford modules and inductively constructing higher dimensional examples

The following examples are all that is needed to build any other Clifford module. The procedure of how to do this is outlined below.

Example 3.

- *Type* $(0,0)$*:* $s = 0$, $V = \mathbb{C}$, $Cv = \overline{v}$, *the chirality is* $\widetilde{\gamma} = 1$
- *Type* $(1,0)$*:* $s = 7$, $V = \mathbb{C}$, *and we have that* $\gamma^1 = e$ *as in Example 1, with* $\widetilde{\gamma} = e$ *also.* $Cv = \overline{v}$
- *Type* $(0,1)$*:* $s = 1$, $V = \mathbb{C}$, $\gamma^1 = \pm i$ *with* $\widetilde{\gamma} = \pm 1$. *Either choice is equivalent and* $Cv = \overline{v}$.
- *Type* $(0,2)$*:* $s = 2$, $V = \mathbb{C}^2$, $\gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$,$\gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *with* $\widetilde{\gamma} = i\gamma^1\gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *We find that* $C \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \overline{v}_2 \\ -\overline{v}_1 \end{pmatrix}$, *by checking the conditions for C against the gammas defined here.*

- *Type* $(1,1)$*:* $s = 0$, $V = \mathbb{C}^2$, $\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *with* $\widetilde{\gamma} = \gamma^1\gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *We find that* $C \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \overline{v}_1 \\ \overline{v}_2 \end{pmatrix}$.
- *Type* $(2,0)$*:* $s = 6$, $\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *with* $\widetilde{\gamma} = i\gamma^1\gamma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. *We find that* $C \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \overline{v}_1 \\ \overline{v}_2 \end{pmatrix}$.

The aim is to take two Clifford modules and product them together in a way to arrive at another Clifford module. This procedure has to be split into two streams, one when at least one of the Clifford modules is of even type ($s = q - p$ mod 8 is even) and the cases when neither of them are even.

Even Case:

Given a (p_1, q_1)-Clifford module $M(p_1, q_1)$ where $s_1 = q_1 - p_1$ mod 8 is even with chirality operators $\widetilde{\gamma}_1$, and another Clifford module $M(p_2, q_2)$ of any type, then the product Clifford module $M(p_1 + p_2, q_1 + q_2)$ is formed by generating it with the following elements:

$$\gamma_1^1 \otimes 1, \gamma_1^2 \otimes, \ldots, \gamma_1^{n_1} \otimes 1, \widetilde{\gamma}_1 \otimes \gamma_2^1, \widetilde{\gamma}_1 \otimes \gamma_2^2, \ldots, \widetilde{\gamma}_1 \otimes \gamma_2^{n_2} \tag{46}$$

These objects act on the space $V_1 \otimes V_2$. Note that $s = ((q_1 + q_2) - (p_1 + p_2))$ mod $8 = (s_1 + s_2)$ mod 8 and so s is even if s_2 is even and odd otherwise and similarly $n = (n_1 + n_2)$ mod 8 and is even if n_2 (equivalently s_2 is even), otherwise it is odd.

The Hermitian inner product on the module is defined to be given by:

$$(v_1 \otimes w_1, v_2 \otimes w_2) = (v_1, v_2)(w_1, w_2) \tag{47}$$

he chirality operator for the new Clifford module can now be determined:

$$\widetilde{\gamma} = i^{s(s+1)/2}(\gamma_1^1 \otimes 1)(\gamma_1^2 \otimes 1)\cdots(\gamma_1^{n_1} \otimes 1)(\widetilde{\gamma}_1 \otimes \gamma_2^1)\cdots(\widetilde{\gamma}_1 \otimes \gamma_2^{n_2}) \tag{48}$$

$$= i^{s(s+1)/2} \underbrace{\gamma_1^1\gamma_1^2\cdots\gamma_1^{n_1}}_{=i^{-s_1(s_1+1)/2}\widetilde{\gamma}_1}(\widetilde{\gamma}_1)^{n_2} \otimes \underbrace{\gamma_2^1\cdots\gamma_2^{n_2}}_{=i^{-s_2(s_2+1)/2}\widetilde{\gamma}_2} \tag{49}$$

$$= i^{(s(s+1)-s_1(s_1+1)-s_2(s_2+1))/2}(\widetilde{\gamma}_1)^{n_2+1} \otimes \widetilde{\gamma}_2. \tag{50}$$

Note that $s = (s_1 + s_2) \bmod 8$ and so $s = s_1 + s_2 - 8\alpha$ for some integer α. So

$$s(s+1) = (s_1 + s_2 - 8\alpha)(s_1 + s_2 - 8\alpha + 1) \tag{51}$$

$$= (s_1 + s_2)(s_1 + s_2 + 1) - 2 \cdot 8\alpha(s_1 + s_2 + 1) + 8^2\alpha^2 \tag{52}$$

$$= (s_1 + s_2)(s_1 + s_2 + 1) + 8 \cdot (8\alpha^2 - 2\alpha(s_1 + s_2 + 1)) \tag{53}$$

$$= s_1(s_1 + 1) + 2s_1s_2 + s_2(s_2 + 1) + 8 \cdot (8\alpha^2 - 2\alpha(s_1 + s_2 + 1)) \tag{54}$$

Consider now

$$s(s+1) - s_1(s_1+1) - s_2(s_2+1) = 2 \cdot s_1s_2 + 8 \cdot (8\alpha^2 - 2\alpha(s_1 + s_2 + 1)) \tag{55}$$

Which results in:

$$(s(s+1) - s_1(s_1+1) - s_2(s_2+1))/2 = s_1s_2 + 4 \cdot (8\alpha^2 - 2\alpha(s_1 + s_2 + 1)) \tag{56}$$

As $i^{4\delta} = 1$ for any integer δ: $i^{(s(s+1)-s_1(s_1+1)-s_2(s_2+1))/2} = i^{s_1s_2}$. The cases are then split into when s_2 is even and odd.

Even s_2

As both s_1 and s_2 are even $s_1s_2 = 4u$ for some integer u, then $i^{s_1s_2} = 1$ also. Yielding

$$\widetilde{\gamma} = (\widetilde{\gamma}_1)^{n_2+1} \otimes \widetilde{\gamma}_2 \tag{57}$$

Recall that $\gamma_1^2 = 1$ and as n_2 is even $(\widetilde{\gamma}_1)^{n_2} = ((\widetilde{\gamma}_1)^2)^{n_2/2} = 1$. Yielding the following for the chirality operator for even s_2:

$$\widetilde{\gamma} = \widetilde{\gamma}_1 \otimes \widetilde{\gamma}_2 \tag{58}$$

Odd s_2

If s_2 is odd then n_2 is odd and so $n_2 + 1$ is even and so $\widetilde{\gamma}_1^{n_2+1} = 1$. Also note that as s_1 is even, so is $s_1 s_2$. The problem is then split into the cases when s_1 is a multiple of 4 and when its just a multiple of 2. If $s_1 = 0, 4$, then $s_1 s_2 = 0, 4s_2$ and $i^{s_1 s_2} = i^0$ or i^{4s_2} and so is equal to 1. When $s_1 = 2, 6$ then $i^{s_1 s_2} = (-1)^{s_2}$ or $(-1)^{3*s_2}$, but s_2 is odd and so is $3s_2$ so $i^{s_1 s_2} = (-1)$. Notice that for $s_1 = 0, 4$ then $\epsilon'' = 1$ and for $s_1 = 2, 6$ then $\epsilon'' = -1$.

So a concisely way to write the formula for the new chirality operator is the following:

$$\widetilde{\gamma} = \begin{cases} \widetilde{\gamma}_1 \otimes \widetilde{\gamma}_2 & s_2 \text{ even} \\ \epsilon_1'' \otimes \widetilde{\gamma}_2 & s_2 \text{ odd.} \end{cases} \tag{59}$$

The real structure now needs to be determined for this new Clifford module. Given that C_1, C_2 are the real structures on $M(p_1, q_1)$ and $M(p_2, q_2)$ respectively, one can check that the following defines a real structure for the product module $M(p_1 + p_2, q_1 + q_2)$.

$$C = \begin{cases} C_1 \otimes C_2 & s_2 \text{ even, } \epsilon_1'' = 1 \\ C_1 \otimes C_2 \widetilde{\gamma}_2 & s_2 \text{ even, } \epsilon_1'' = -1 \\ C_1 \otimes C_2 & s_2 \text{ odd, } \epsilon' = 1 \\ C_1 \widetilde{\gamma}_1 \otimes C_2 & s_2 \text{ odd, } \epsilon' = -1. \end{cases} \tag{60}$$

Thus any Clifford module $M(p,q)$ can be constructed by taking products of $M(0,2)$ and $M(2,0)$ and then product with either $M(1,0)$ or $M(0,1)$ on the right, to achieve any (p,q)-type.

Two constructions for the fuzzy sphere are presented later, one via a $M(0,3)$ Clifford module and one via a $M(1,3)$ Clifford module. In order to relate them to each other, the following construction between the product of two odd Clifford modules will be useful, as it will allow $M(1,3)$ to be written as the product of $M(1,0) \cdot M(0,3)$.
Odd Case:
This product is defined via a trick. First pick a Clifford module $M(p_1+1,q_1)$ from which we can identify $M(p_1,q_1)$ by ignoring the extra generator t (where $t^2=1$). The complete list of generators for $M(p_1+1,q_1)$ is then

$$t, \gamma_1^1, \gamma_1^2, \ldots, \gamma_1^n. \tag{61}$$

The chirality operator, Γ, for $M(p_1+1,q_1)$ is then given by the usual definition. To proceed, define the product $M(p_1+p_2+1,q_1+q_2) = M(p_1+1,q_1)\cdot M(p_2,q_2)$ as defined above for the even cases. The module $M(p_1+p_2,q_1+q_2)$ can be defined from $M(p_1+p_2+1,q_1+q_2+1)$ by ignoring the generator $t\otimes 1$. Thus for $M(p_1+p_2,q_1+q_2)$ the generators are the following:

$$\gamma_1^1\otimes 1,\ldots,\gamma_1^{n_1}\otimes 1,\Gamma\otimes\gamma_2^1,\ldots,\Gamma\otimes\gamma_2^{n_2} \tag{62}$$

Note that Γ still has the generator t in it. This procedure is best understood via an example:

Example 4. *The product between $M(1,0)$ and another odd Clifford module $M(p,q)$ is given by identifying $M(1,0)$ with the restriction of $M(2,0)$, given by the representation in*

example 3. Setting $t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and recalling that* $\Gamma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ *so that the new module* $M(p+1,q)$ *is generated by*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1_{(n_2-1)/2} = \begin{pmatrix} 0 & 1_{(n_2-1)/2} \\ 1_{(n_2-1)/2} & \end{pmatrix}, \qquad \Gamma \otimes \gamma_2^i = \begin{pmatrix} 0 & i\gamma_2^i \\ -i\gamma_2^i & 0 \end{pmatrix}. \quad (63)$$

The chirality operator is then given by

$$\widetilde{\gamma} = i^{s(s+1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma^{n_2} \otimes \gamma_2^1 \cdots \gamma_2^{n_2}. \quad (64)$$

Note that $\Gamma^2 = 1$ *as it is a chirality operator for a* $(2,0)$*-Clifford module, and so* $\Gamma^{n_2} = \Gamma$ *as* n_2 *is odd. Yielding*

$$\widetilde{\gamma} = i^{s(s+1)/2} i^{-s_2(s_2+1)/2} (-i) \begin{pmatrix} \widetilde{\gamma}_2 & 0 \\ 0 & -\widetilde{\gamma}_2 \end{pmatrix} \quad (65)$$

Writing $s = s_1 + s_2 - 8\cdot\alpha = 7 + s_2 - 8\cdot\alpha$ *it is found that* $s(s+1) - s_2(s_2+1) = s_1(s_1+1) + 2s_1s_2 - 8\cdot\delta = 56 + 14s_1 - 8\cdot\delta$*. Recall that* $i^4 = 1$ *so that* $i^{s(s+1)/2} i^{-s_2(s_2+1)/2} = i^{28+7s_2} = i^{7s_2} = (-i)^{s_2}$*. Combining this above, it is found that*

$$\widetilde{\gamma} = (-i)^{s_2+1} \begin{pmatrix} \widetilde{\gamma}_2 & 0 \\ 0 & -\widetilde{\gamma}_2 \end{pmatrix} \quad (66)$$

Considering all possibilities for s_2 *and using that* $s_1 = 7$ *it is found that* $(-i)^{s_2+1} = -\epsilon''$ *so that:*

$$\widetilde{\gamma} = -\epsilon'' \begin{pmatrix} \widetilde{\gamma}_2 & 0 \\ 0 & -\widetilde{\gamma}_2 \end{pmatrix} \quad (67)$$

The real structure then takes the following form:

$$C = \begin{cases} \widetilde{C}_1 \otimes C_2 & \epsilon'' = 1 \\ \widetilde{C}_1\gamma_1 \otimes C_2\gamma_2 & \epsilon'' = -1 \end{cases} \tag{68}$$

Now comes the key definition of this thesis:

Definition 29. A type (p, q) matrix geometry is a finite real spectral triple $(s, \mathcal{H}, \mathcal{A}, \Gamma, J, D)$ formed by the following product of a (p, q)-Clifford module and a type $(0, 0)$ matrix geometry.

- $s = q - p \bmod 8$
- $\mathcal{A} = \mathcal{A}_0$
- $\mathcal{H} = V \otimes \mathcal{H}_0$ with inner product $\langle v \otimes m, v' \otimes m' \rangle = (v, v')\langle m, m \rangle$
- $\rho(a)(v \otimes m) = v \otimes \rho_0(a)m$
- $\Gamma = \widetilde{\gamma} \otimes 1$
- $J = C \otimes J_0$
- D is any operator allowed by the axioms of a finite real spectral triple

The Clifford module is required to be irreducible if s is even and the eigenspaces of the chirality operator $\widetilde{\gamma}$, $V_\pm \subset V$ to be irreducible if s is odd.

An investigation into what form D can take is outlined for the above setup.

2.2.5 *Matrix Dirac Operators*

Lemma 1. *The Dirac operator for a finite dimensional spectral triple can be written as*

$$D = \theta + \epsilon' J\theta J^{-1} \tag{69}$$

where $\theta^* = \theta$ *and* $[\theta, J\rho(a)J^{-1}] = 0$ *for all* $a \in \mathcal{A}$. *Also* $\{\theta, \Gamma\} = 0$ *when* s *is even and* $[\theta, \Gamma] = 0$ *when* s *is odd.*

Proof. The proof of this lemma is explicitly spelled out in Barrett (2015) [46] □

For a given (p, q) matrix geometry[7] $End_{\mathbb{C}}(\mathcal{H}) \cong End_{\mathbb{C}}(V) \otimes End_{\mathbb{C}}(\mathcal{H}_0)$ so θ can be written as $\theta = \sum_i \omega_i \otimes X_i$, where the $\omega_i \in End_{\mathbb{C}}(V)$ form a linearly independent set and $X_i \in End_{\mathbb{C}}(\mathcal{H}_0)$ satisfy $[X_i, J_0\rho_0(a)J_0^{-1}] = 0$ for all $a \in \mathcal{A}_0$. The condition that $\theta^* = \theta$ requires that ω_i and X_i are either both Hermitian or anti-Hermitian.

The ω_i's

Let Ω be the $\mathbb{R}$-algebra generated by the matrices γ^a in the (p, q)-Clifford module. Ω can be decomposed in the following way: $\Omega = \Omega_- \oplus \Omega_+$, where Ω_- is the subalgebra generated by odd products of γ^a and Ω_+ is generated by the even products. Recall that for even s, V had to be an irreducible Clifford module, in that the map $c : CL(p, q) \to End_{\mathbb{R}}(V)$ or $End_{\mathbb{C}}(V)$ exists. In either case $\Omega \otimes_{\mathbb{R}} \mathbb{C} = End_{\mathbb{C}}(V)$ by the irreducibility of the Clifford module.

Recall that there are different conditions on θ, and therefore the ω_i, when s is even and s is odd. For even s that $\gamma^i\widetilde{\gamma} = -\widetilde{\gamma}\gamma^i$, and so it is found that the ω_i need to be taken from $\Omega_- \otimes_{\mathbb{R}} \mathbb{C}$ in order for the conditions $\{\theta, \Gamma\} = 0$, which reduces to $\omega_i\widetilde{\gamma} = -\widetilde{\gamma}\omega_i$, to be satisfied. For odd s it is required that $[\theta, \Gamma] = 0$, which is just $[\omega_i, \widetilde{\gamma}] = 0$, which is satisfied by any element of $\Omega \otimes_{\mathbb{R}} \mathbb{C}$. In either case the complex coefficients can be absorbed into the X_i, resulting in the $\omega_i \in \Omega_-$ when s is even and the $\omega_i \in \Omega$ when s is odd. Recall that γ^a is Hermitian if $(\gamma^a)^2 = 1$ and it is anti-Hermitian if $(\gamma^a)^2 = -1$.

The X_is

To examine the possibilities for the $X_i \in End_{\mathbb{C}}(\mathcal{H}_0)$ first realise that $\mathcal{H}_0 \subseteq M_n(\mathbb{C}) \cong \mathbb{C}^n \otimes (\mathbb{C}^n)^*$, and due to $\mathcal{H}_0$ being finite dimensional $End_{\mathbb{C}}(\mathcal{H}_0) \cong End(\mathbb{C}^n) \otimes End_{\mathbb{C}}((\mathbb{C}^n)^*)$. Regarding elements of $\mathbb{C}^n$ as column vectors and elements of $(\mathbb{C}^n)^*$ as row vectors,

7 For this decomposition to hold, the components of $\mathcal{H} = V \otimes \mathcal{H}_0$ need to be finitely-generated projective modules [78, Chp 2. sec. 4.4, prop 4.], which both V and $\mathcal{H}_0$ are.

the elements of $End(\mathbb{C}^n)$ are then left acting matrices and $End((\mathbb{C}^n)*)$ are right acting matrices so that (in the notation of [46])

$$X_i(m) = \sum_j Y_{ij} m Z_{ij} \tag{70}$$

for a finite set of matrices $Y_{ij}, Z_{ij} \in M_n(\mathbb{C})$.

The condition that $[X_i, J_0\rho_0(a)J_0^{-1}] = 0$ becomes $\sum_j Y_{ij} m[Z_{ij}, a] = 0$, in which a set of linearly independent matrices can be chosen for Y_{ij} and the condition becomes $[Z_{ij}, a] = 0$.

If $\mathcal{H}_0$ is an irreducible bimodule then because Z_{ij} commutes with the right action of the algebra, by Shurs Lemma each Z_{ij} acts as a $\mathbb{C}$-multiple of the identity. Therefore $X_i(m) = \sum_j Z_{ij} Y_{ij} m$ in that case.

If $\mathcal{H}_0$ is a direct sum of irreducible bimodules

$$\mathcal{H}_0 = \bigoplus_{i,j} H_{ij} \tag{71}$$

where i labels the irreducibles for the left action and j for the right action of $\mathcal{A}_0$. Then as Z_{ij} has to commute with the right action of $\mathcal{A}_0$, Z_{ij} acts as a scalar on each irreducible right module and thus can only permute the left ρ_i-modules for each right action (specified by j).

In the case when only one of the H_{ij} is nonzero for each i, so that $\mathcal{H}_0$ resembles a block permutation matrix but where the identity matrices are replaced by irreducible bimodules. Then the action of the Z_{ij} is block diagonal (to preserve the block decomposition) and can also be represented by a left acting matrix Z'_{ij} where the diagonal blocks are permuted. So again $X_i(m) = \sum_j Y_{ij} Z'_{ij} m$. This is best understood via an example.

Example 5. *Let $\mathcal{H}_0$ have the form*

$$\mathcal{H}_0 = \begin{pmatrix} 0 & H_{12} & 0 \\ H_{21} & 0 & 0 \\ 0 & 0 & H_{33} \end{pmatrix} \tag{72}$$

where each H_{ij} is an irreducible bimodule and each H_{ij} are inequivalent. Then if

$$Z = \begin{pmatrix} a\mathbb{I} & 0 & 0 \\ 0 & b\mathbb{I} & 0 \\ 0 & 0 & c\mathbb{I} \end{pmatrix} \tag{73}$$

the right action mZ for $m \in \mathcal{H}_0$ can be written as a left action $Z'm$ where

$$Z' = \begin{pmatrix} b\mathbb{I} & 0 & 0 \\ 0 & a\mathbb{I} & 0 \\ 0 & 0 & c\mathbb{I} \end{pmatrix} \tag{74}$$

Let K_i denote the matrix $\sum_{ij} Y_{ij} Z'_{ij} \in M_n(\mathbb{C})$. Then $\theta(v \otimes m) = \sum_i (\omega_i v) \otimes (K_i m)$ and so the Dirac operator has the form:

$$D = \sum_i \omega_i v \otimes K_i m + (\epsilon')^{\alpha_i + 1} m K_i^* \tag{75}$$

where $\alpha_i \in \mathbb{N}$ is the number of γ^a in ω_i.

2.2.6 *Fuzzy Spaces*

Definition 30. Let V be a Clifford module of type (p, q) with chirality operator γ and real structure C. For $p + q$ even let V be irreducible and for $p + q$ odd let the chiral subspaces $V_\pm$ be irreducible. A *fuzzy space* is a real spectral triple with the following objects

- $\mathcal{H} = V \otimes M_n(\mathbb{C})$
- $\Gamma = \widetilde{\gamma} \otimes 1$
- $\mathcal{A} = M_n(\mathbb{C})$
- $J = C \otimes J_0$

where the inner product on $\mathcal{H}$ is defined by:

$$\langle v_1 \otimes m_1, v_2 \otimes m_2 \rangle = (v_1, v_2)\mathrm{Tr}(m_1^* m_2) \tag{76}$$

and the action of the algebra is just my multiplication on $M_n(\mathbb{C})$, i.e.

$$\rho(a)(v \otimes m) = v \otimes (am). \tag{77}$$

The actions of Γ and J on an element of $\mathcal{H}$ are as follows:

$$\Gamma(v \otimes m) = \widetilde{\gamma} v \otimes m, \qquad J(v \otimes m) = Cv \otimes m^* \tag{78}$$

And the final object is the Dirac operator, D, takes the following forms:
$\underline{\epsilon' = 1}$

$$D = \sum_i \tau_i \otimes \{H_i, \cdot\} + \sum_j \alpha_i \otimes [L_i, \cdot] \tag{79}$$

where α_j and τ_i are a product of an odd number of γ^a, τ_i and H_i are both Hermitian and α_j and L_j are both anti-Hermitian.
$\underline{\epsilon' = -1}$

$$D = \sum_i \tau_i^+ \otimes [H_i, \cdot] + \sum_j \alpha_j^+ \otimes \{L_i, \cdot\} + \sum_k \tau_k^- \otimes \{H_i, \cdot\} + \sum_l \alpha_l^- \otimes [L_l, \cdot] \tag{80}$$

where now τ_i^+, α_j^+ are even products of γ^a and τ_k^-, α_l^- are odd products. Also $\alpha_j^+, \alpha_l^-, L_i, L_l$ are all anti-Hermitian and $\tau_i^+, \tau_k^-, H_i, H_k$ are all Hermitian.

2.3 KEY EXAMPLES OF FUZZY SPACES

As mentioned in the introduction the term fuzzy space can refer to a more general object that outlined in this thesis. There are many constructions in the literature that describe noncommutative analogues of ordinary manifolds, that do not necessary lie within the framework of this thesis. For instance there are matrix descriptions of the 2-sphere [49, 51, 52, 53, 54, 57, 55, 79, 56, 80, 60, 81], the complex projective spaces (which often include the sphere also) [59, 58, 82, 83, 84] and fuzzy Riemann surfaces [85] to list a few. Even more examples of noncommutative geometries exist if you look for noncommutative but not finite analogues, such as noncommutative tori [86], the quantum pillow [87] and quantum homogeneous spaces [88, 89, 90].

Many of these constructions do not construct a version of the Dirac operator, and if they do, it need not satisfy the axioms laid out above. A discussion of two other approaches for the fuzzy sphere by Watamura-Watamura [52, 53, 57] and Balachandran et al [54, 60] are discussed to understand why they do or do not fit in the fuzzy space picture. The tools developed in chapter 3 are spectral in nature and in theory can be used to analyse any geometry with a fuzzy analogue of a differential operator. However, the geometric interpretation would become unclear for geometries that stray too far from the duality of Connes between spin manifolds and spectral triples.

2.3.1 *The Fuzzy Sphere*

The fuzzy sphere is one of the simplest examples of a noncommutative geometry and provides key insights into their construction. In this thesis we are concerned with noncommutative geometry via the finite real spectral triple description. In this approach, the fuzzy sphere belongs to a exclusive club of finite spectral triples that approximate in some manner a continuum space. The only other example known is that of the fuzzy torus defined in [47] which is introduced below.

We want to view fuzzy spaces as approximations to ordinary manifolds, where the algebra of functions is deformed to be noncommutative. We also want to view the operator D as the fuzzy version of the Dirac operator on the sphere, which requires us to understand the construction of a spinor bundle and the Dirac operator on the ordinary 2-sphere.

We will investigate the more general differential geometric framework in which the 2-sphere fits in a later section (as a homogeneous space and coadjoint orbit of a Lie group). However for the intended purposes of this chapter we will omit some details.

In order to define the canonical spectral triple for the ordinary 2-sphere we need to construct the smooth functions $C^\infty(S^2)$ and the spinor bundle. We start by describing $C^\infty(S^2)$ in a manner than offers insight into the fuzzy sphere. Consider the description of the sphere as the subspace of the three dimensional Euclidean space satisfying the condition that $x_1^2 + x_2^2 + x_3^2 = 1$. Take $\mathcal{P}$ to be the algebra of polynomials in the coordinates x^a for $a = 1, 2, 3$, with the product structure given by pointwise multiplication. Let $\mathcal{I}$ be the ideal generated by $\delta_{ab}x^a x^b - 1$, then the quotient space $\mathcal{A} = \mathcal{P}/\mathcal{I}$ is dense in the algebra $C(S^2)$, as the function $f(x^a) = \sum_a x^a$ separates points, the closure of $\mathcal{A}$ under the sup-norm is again a subalgebra and it contains the constant functions. We can write any element of $\mathcal{A}$ in the following form:

$$f(x^a) = f_0 + f_a x^a + \frac{1}{2} f_{ab} x^a x^b + \frac{1}{3!} f_{abc} x^a x^b x^c + \cdots \tag{81}$$

where because $x^a x^b = x^b x^a$ each coefficient $f_{a_1 a_2 \ldots}$ is symmetric in each pair of indices. The condition that $\delta_{ab}x^a x^b = 1$ states that $f_{a_1 a_2 \cdots}$ is trace-free (i.e. that we have $\delta_{a_i a_j} f_{a_1 a_2 \cdots} = 0$ for any a_i, a_j pair). So for the 2-sphere the space $\mathcal{A} = \mathcal{P}/\mathcal{I}$ can be considered its algebra of functions.

There are a number of possible ways to replace this algebra of functions with a finite dimensional algebra. If the functions are approximated by truncating their series expansion after the n^{th} products, i.e.

$$f(x^i) = f_0 + f_a x^a + \cdots + f_{a_1 a_2 \ldots a_n} x^{a_1} \ldots x^{a_n} \tag{82}$$

and take all the functions of this type, this results in a n-dimensional vector space of the entries $f_0, \ldots, f_{a_1 a_2 \ldots a_n}$. In order to turn this vector space into an algebra requires some thought. The normal product of two elements f, g, would obviously be outside of this set of objects as it would require terms such as $x^{a_1} x^{a_2} \ldots x^{a_n} x^{a_{n+1}}$ etc. Looking at the case when $n = 1$, one could define a product to be $f \cdot g = f_0 g_0 + \sum_{a=1}^{3} f_a g_a x^a$, this would turn the vector space into an algebra. However, this algebra can be identified with four copies of $\mathbb{C}$, which is exactly the algebra of functions at four discrete points. This product can be extended to any value for n in the straight forward manner and still only results in an algebra which consists of the functions on a set of discrete points. This is unappealing because these point will not be invariant under rotations and thus we lose a defining property of the sphere.

There is a way to preserve this invariance under rotations (i.e. invariant the action of $SO(3)$), done by making the algebra noncommutative. Start by taking the n-dimensional generators of the Lie algebra of $SO(3)$, $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, and denote them by L_a. Then set $x^a = \kappa L_a$, where κ is a constant that is defined by requiring $\sum_a (x^a)^2 = 1$. Note that $\sum_a (x^a)^2 = \sum_i \kappa^2 (L_a)^2 = \kappa^2 C$ where C is the quadratic Casimir element in the n-dimensional representation of $\mathfrak{su}(2)$. This is equal to $\frac{n^2-1}{4}$ for the generators that satisfy $[L_a, L_b] = i\epsilon_{abc} L_c$ [91]. So that $\sum_i (x^i)^2 = \kappa^2 (n^2 - 1)/4$. So taking $\kappa = 2/\sqrt{n^2 - 1}$.

Notice that the coordinates no longer commute, we have $\left[x^1, x^2\right] = i\kappa x^3$, and the cyclic permutations. So the notion of a point (x^1, x^2, x^3) vanishes as one can never know all three entries at the same time, i.e. it is impossible to simultaneously diagonalise each of the x^i. However, by considering the 2-dimensional representation, which are the Pauli matrices divided by 2, each generator has 2 eigenvalues. So

picking a single coordinate $x^i = \kappa\sigma^i/2 = \sigma^i/\sqrt{3}$ it is possible to get two eigenvalues namely $\pm\frac{1}{\sqrt{3}}$. This is interpreted as being able to only distinguish which hemisphere of the sphere we lie in. As we increase the dimension of the representation we get more eigenvalues for each of the generators, which means we can narrow down the zone we lie in. Which gives credence to the name of fuzzy sphere, as the spaces we end up with are like the points of a sphere have been smeared together in a certain fashion.

Note that as $n \to \infty$ then $\kappa \to 0$, so the coordinates in the limit commute and the normal commutative sphere is recovered. This interpretation gives us a natural picture of what we mean by a 'quantum geometry'. However this limit is difficult to define rigorously, with it being formalised in [92].

The above construction is the canonical viewpoint, however, there is another interpretation which is arguably more enticing as it imposes a cutoff on the spherical harmonics and in turn the energy states, the details of this treatment can be found in [45]. Going back to the algebra of coordinates, $C(\mathbb{S}^2)$, it can decomposed in to a direct sum of irreducible representations[8] of $su(2)$:

$$C(\mathbb{S}^2) \simeq \bigoplus_{l=0}^{\infty} V_l, \tag{83}$$

where V_l is the vector space underlying the irreducible representation of $su(2)$ with the highest weight $l \in \mathbb{N}$. The vector space V_l is spanned by the spherical harmonics $Y_{l,m}$. A cutoff in the energy spectrum an then be imposed by ignoring all but the first $n+1$ representations in the decomposition of $C(\mathbb{S}^2)$. Thus, in the fuzzy sphere's spectral triple the *fuzzy spherical harmonics*[9] $\hat{Y}_{l,m}$ are taken as the generators of $M_{n+1}(\mathbb{C})$, where[10] $l < n$. The algebra is then decomposed into

$$\mathcal{A}_n \simeq \bigoplus_{l=0}^{n} V_l, \tag{84}$$

8 Exactly how to arrive at this result is described in the appendix A.

9 These are matrix versions of the spherical harmonics, there precise construction can be found in [45].

10 Note that the reason $l < n$ not $l < n+1$ is because the index l starts at zero. So there are still $n+1$ generators for $M_{n+1}(\mathbb{C})$

where V_l is as above. The space V_l is spanned by the fuzzy spherical harmonics $\hat{Y}_{l,m}$ for $m = -l, \ldots, l$. Thus a fuzzy sphere can be viewed as having a maximum angular momentum and therefore an energy cutoff, which can be recast as a minimal renderable distance, i.e. a Planck length. The implications of a Planck length being a natural outcome of requiring the underlying space to be finite and noncommutative is a very appealing property, and the idea is that a noncommutative analogue to a spacetime will provide a good model for quantum gravity.

However, the algebra of functions does not define a metric or any of the other structures required for a spectral triple. The usual metric desired on the sphere is the *round metric*. This can be defined as the metric inherited from the embedding into $\mathbb{R}^3$. Given the normal vector $N = (x_1, x_2, x_3)$ for the corresponding point of S^2, the flat (inverse) metric on $\mathbb{R}^3$ can be pulled back to the sphere to attain the metric[11]: $g_{ab} = \delta_{ab} - N_a N_b$, producing:

$$g^{-1}(x_1, x_2, x_3) = \begin{pmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_2 x_1 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_3 x_1 & -x_3 x_2 & x_1^2 + x_2^2 \end{pmatrix}. \tag{85}$$

This inverse metric is degenerate and has rank 2, and is annihilated by the vector (x_1, x_2, x_3). Thus is cannot be inverted it in this form. Using the standard spherical parametrisation:

$$x_1 = \cos(\phi)\sin(\theta) \tag{86}$$

$$x_2 = \sin(\phi)\sin(\theta) \tag{87}$$

$$x_3 = \cos(\theta) \tag{88}$$

11 The details of this method can be found in Hawking and Ellis [93]

for $0 \leq \phi < 2\pi$ and $0 \leq \theta < \pi$, the usual formula for the round inverse metric can be attained by the formula $g^{-1}(\theta,\phi) = Jac(\theta,\phi)g^{-1}(x_i(\theta,\phi))Jac(\theta,\phi)^T$ which now has full rank and can be inverted to give

$$g(\theta,\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}. \tag{89}$$

which is the usual expression for the round metric for the 2-sphere.

The reason the Cartesian expression for the inverse metric is introduced is that it is easier to compare the fuzzy Dirac operator to the ordinary Dirac operator when making use of the embedding in $\mathbb{R}^3$. The following method for getting the induced Dirac operator on a hypersurface (a subspace of codimension 1) in Euclidean space $(\mathbb{R},\delta)$ is a very brief and to the point method taken from [94] and calculations for the sphere are given.

It is shown in [94] that if an oriented hypersurface is isometrically immersed in Euclidean space then the spinor bundle is trivial. Thus for such a space M the space of square-integrable sections of the spinor bundle, $L^2(S)$, can be viewed as $L^2(M) \otimes \mathbb{C}^k$, for some power of k depending on the dimension of the Euclidean space. The method of constructing the Dirac operator then begins with taking the polynomial that defines the hypersurface, $f(x^i) = 0$. First, for an embedding in $\mathbb{R}^n$, take n Dirac matrices, $\{\gamma_a\} \in End(\mathbb{C}^k) \cong M_k(\mathbb{C})$ where $k = n/2$ for even n and $k = (n-1)/2$ for odd n, such that they satisfy the relation $\gamma_a\gamma_b + \gamma_b\gamma_a = -2\delta_{ab}$. Note that for this definition, $\gamma_a^2 = -1$ Next, find the unit normal vectors to the surface N. This can be done by using the formula $N = \frac{\nabla f}{|\nabla f|}$. Then the formula for the Dirac operator, taken from [94], is as follows:

$$D = \sum_{i,j=1}^{3} (\gamma_i\gamma_j + \delta_{ij})N_i\partial_j + \frac{1}{2}\mathrm{div}(N), \tag{90}$$

where N_i are the ith components of the normal vector and

$$\mathrm{div}(N) = \sum_{i,j=1}^{3} \left(\delta_{ij} - N_i N_j\right) \partial_i N_j. \tag{91}$$

This Dirac operator is the one associated to the metric induced from the embedding into Euclidean space.

For the sphere, the embedding is into $\mathbb{R}^3$, so three gamma matrices are taken $\gamma^i \in M_2(\mathbb{C})$, which act on 2-dimensional spinors $\mathbb{C}^2$. The 2-sphere is defined by $\sum_{i=1}^{3} (x^i)^2 = 1$, thus the normal vectors are given by $N^i = x^i$. It was shown above that the induced metric from this embedding is the round metric that is desired, and thus we can calculate the Dirac operator using eq. (90). First, note that $\partial_i N_j = \delta_{ij}$ and therefore:

$$\mathrm{div}(n) = \sum_{i,j=1}^{3} \delta_{ij}\delta_{ij} - N_i N_j \delta_{ij} = \underbrace{\sum_{i=1}^{3} \delta_{ii}}_{=3} - \underbrace{\sum_{i=1}^{3} x_i^2}_{=1} = 2. \tag{92}$$

Making use of this result in (90), the Dirac operator becomes:

$$D = \sum_{i,j=1}^{3} (\gamma_i \gamma_j + \delta_{ij}) x_i \partial_j + 1 = \sum_{i \neq j=1}^{3} \gamma_i \gamma_j x_i \partial_j + \sum_{i=1}^{3} (\gamma_i^2 + 1) x_i \partial_j + 1 \tag{93}$$

Here the fact that $\gamma_i^2 = -1$ is used and thus the middle term vanishes in general to give:

$$D = \sum_{i \neq j=1}^{3} \gamma_i \gamma_j x_i \partial_j + 1 \tag{94}$$

Using that $\gamma_i \gamma_j = -\gamma_j \gamma_i$ for $i \neq j$, the final form for the Dirac operator on $\mathbb{S}^2$ is:

$$D_{\mathbb{S}^2} = \sum_{i<j=1}^{3} \gamma_i \gamma_j (x_i \partial_j - x_j \partial_i) + 1 \tag{95}$$

This Dirac operator has eigenvalues $\mathbb{Z}\backslash\{0\}$ with eigenvalues l appearing with multiplicity $2|l|$ (see [94] for a proof of this). Thus the spectral triple for a 2-sphere is given by $(\mathcal{A} = \oplus_{l=1}^{\infty} V_l, L^2(\mathbb{S}^2) \otimes \mathbb{C}^2, D_{\mathbb{S}^2})$

We are now ready to present the spectral triple for the fuzzy sphere. There are a number of ways to express a spectral triple for the fuzzy sphere. The most noteworthy approaches for this thesis are those by Grosse and Prešnajder [51] who constructed the Dirac operator for the fuzzy sphere and in [45] it was shown to be part of a formal spectral triple. The other approach is given by Barrett in [46], which is similar to the other approaches however with a different Clifford module structure and arrives at a symmetric spectrum and a K0 dimension of 2 that matches the ordinary 2-spheres.

Both approaches take the algebra to be the n-dimensional matrix algebra $\mathcal{A}_n = M_n(\mathbb{C})$, which is viewed as being decomposed into $A_{n+1} \cong V_j \otimes V_j^* = \oplus_{l=0}^{2j} V_l$ as above (note that j can take integer and half integer values as it is equal to $N/2$ but l takes values in the non-negative integers).

For the Grosse-Prešnajder approach (see [45] for further details) the fact that spinor bundle for the 2-sphere is trivial for the 2-sphere is used, and the Hilbert space required to be $\mathcal{A}_n \otimes \mathbb{C}^2$, where $\mathbb{C}^2$ is equipped with a Clifford module structure of type $(0,3)$. The Dirac operator is then taken to be:

$$D_{G-P} = 1 \otimes 1 + \sum_{i=1}^{3} \sigma^i \otimes [L_i, \cdot], \tag{96}$$

where L_i are the n-dimensional representation of the real $su(2)$ generators. The right action needed for the commutator, is defined to be given by $J(v \otimes m) = \sigma^2 v^\dagger \otimes m^*$, where σ^i are the representation of the generators of $su(2)$ in the representation corresponding to $j = 1/2$, i.e. they are related to the Pauli matrices. It was shown that this is a real spectral triple but has no grading, Γ, and that the eigenvalues of the Dirac operator are $\pm l$ with multiplicity $2l$ for $l = 1, 2, \ldots N$ and eigenvalue $+(N+1)$ with multiplicity $2(N+1)$ in [45]. Note that this spectrum is asymmetric due to the lack of even grading.

The method implemented by Barrett [46], enriches the $(0,3)$ geometry to a $(1,3)$ geometry, where the new Hermitian generator is denoted by γ_0. In which the K0 dimension would be $s = 3 - 1 = 2$, matching the 2-sphere. Again the algebra is taken to be $\mathcal{A}_n$ and now the Hilbert space is given by $\mathcal{A}_n \otimes \mathbb{C}^4$ with $\mathbb{C}^4$ a type $(1,3)$-Clifford module. The Dirac operator for this fuzzy sphere is taken to be:

$$D_B = \gamma_0 + \sum_{i<j=1}^{3} \gamma_0\gamma_i\gamma_j \otimes \left[L_{ij}, \cdot\right] \tag{97}$$

where L_{ij} are the generators of $so(3)$ that satisfy:

$$[L_{jk}, L_{lm}] = \delta_{kl}L_{jm} - \delta_{km}L_{jl} - \delta_{jl}L_{km} + \delta_{jm}L_{kl} \tag{98}$$

and are antisymmetric in their indices. Note that the product $\gamma_0\gamma_i\gamma_j$ is anti-Hermitian and an odd product so this definition of the Dirac operator agrees with the Dirac operator definitions in eq. (79). The real structure and chirality operator are as given in definition 30.

It is possible to arrive back at the Grosse-Prešnajder Dirac operator from the Barrett operator shown in [46]. This is done by expressing the $(1,3)$ Clifford module as arriving from the product of a $(0,3)$ Clifford module and a $(1,0)$ Clifford module, as outlined in section 2.2.4. Let σ^a be the generators of the $(0,3)$ Clifford module then we have that:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^a = \begin{pmatrix} 0 & i\sigma^a \\ -i\sigma^a & 0 \end{pmatrix} \tag{99}$$

Thus the Barrett Dirac operator can be expressed as:

$$D_B = \begin{pmatrix} 0 & D_{G-P} \\ D_{G-P} & 0 \end{pmatrix}. \tag{100}$$

The link between these Dirac operators and the commutative 2-sphere's Dirac operator is more clearly seen when the commutative 2-spheres Dirac operator is expressed as embedded from $\mathbb{R}^3$ as described above. To do this, note that the vector

fields $X_{ij} = x_i\partial_j - x_j\partial_i$ satisfy the same Lie algebra relations as the elements $[L_{ij}, \cdot]$. Under the notion that the X_{ij} are the infinite matrix representation of the Lie algebra generators, it can be seen that these Dirac operators all take a similar form. The formal version of this statement is a hard question and comparing fuzzy spaces to their infinite counterparts is an active area of research - see for instance [92, 95].

It should be noted that there are other matrix geometry constructions, that are complete with an operator emulating the Dirac operator. The notable examples are those by Watamura-Watamura [52, 53, 57] and Balachandran et al [54, 60].

The procedure of Watamura and Watamura enforces the existence of a matrix analogue of the chirality operator for the sphere. They then proceed to investigate the possible operators that anti-commute with it. The chirality operator they find in [52] is of the form:

$$\gamma_W = \frac{1}{\mathcal{N}}(\sum_i \sigma^i \otimes x_i + \frac{\kappa}{2}), \tag{101}$$

where the x_i are the generators of the matrix algebra that satisfy $[x_i, x_j] = i\kappa\epsilon_{ijk}x_k$ and κ is defined as above. The $\mathcal{N}$ is a normalisation factor to ensure that $\gamma_W^2 = 1$. This operator limits to the continuum chirality operator $\gamma = \frac{1}{r}\sum_i \sigma_i \otimes x_i$ as described in [96]. However this operator does not commute with the action of the algebra $\gamma_W a \neq a\gamma_W$. A modification was proposed in [53] by using the right action of the matrix algebra instead of the left,i.e.

$$\gamma_W^o = \frac{1}{\mathcal{N}}(\sum_i \sigma^i \otimes x_i^o - \frac{\kappa}{2}). \tag{102}$$

The modified chirality operator however does not commute with the right action of the algebra. Thus there does not exist a real structure J that commutes with the chirality operator[12], breaking condition (iv) of the axioms in definition 19.

A matrix analogue of the Dirac operator is then found by considering all operators that anti-commute with the chirality operator γ_W^o. Notably, this is a property of the Dirac operator and chirality operator for the continuum 2-sphere, but not what is

12 As $\epsilon'' = 1$ for a $(0,3)$ Clifford module, we would require $J\gamma_W = \gamma_W J$.

prescribed by condition (v) of definition 19 for a $(0,3)$ Clifford module. The operator is found to be:

$$D_W = \frac{i}{l\kappa}\gamma^o_W \epsilon_{ijk}\sigma_i x^o_j x_k. \tag{103}$$

This Dirac operator does not fit into the framework of eq. (79) or eq. (80). However, it does strike a resemblance to the chiral rotation of continuum Dirac operator. If we let $\not{D}_1 = i\sum_{ijk}\epsilon_{ijk}\sigma_i(x_j\partial_k) + 1$ like in eq. (90), then we can form a different expression of the Dirac operator as $\not{D}_2 = i\gamma\not{D}_1 = i\epsilon_{ijk}\sigma_i x_j \mathcal{J}_k$, where $\mathcal{J}_k = \epsilon_{abk}(x_a\partial_b) + \sigma_k/2$. It is shown in [57] that Watamura Dirac operator D_W is a matrix analogue of $\not{D}_2$. However, the Watamura Dirac operator D_W is not a chiral rotation of the Grosse-Presnadjer operator, which is the matrix analogue of $\not{D}_1$. As the Watamura fuzzy sphere is not a fuzzy space as meant in this thesis, it is not explored. It is however an interesting object of study, and whether or not it could be included in some generalisation of fuzzy spaces, such as twisted spectral triples [97, 98], is worth investigating.

Despite the construction being different to the usual methods, the Balachandran method as outlined in [60] produces the same Dirac operator as the Grosse-Presnadjer. Starting from a representation of the Ginsperg-Wilson algebra, two operators are constructed. One of which limits to the continuum Dirac operator and is of the same form as the Grosse-Presnadjer operator, the other limits to the continuum chirality operator. However, these operators do not anti-commute and thus at the matrix geometry level, these operators do not satisfy the axioms for a fuzzy space.

As such, for the fuzzy sphere only the Barrett and Grosse-Presnadjer Dirac operators will be considered in the following chapters.

2.3.2 *The Fuzzy Torus*

A finite real spectral triple for the fuzzy torus has recently been developed by John Barrett and James Gaunt [48]. The construction for a fuzzy torus is a lot more subtle that for the fuzzy sphere and requires careful attention. A brief introduction to the

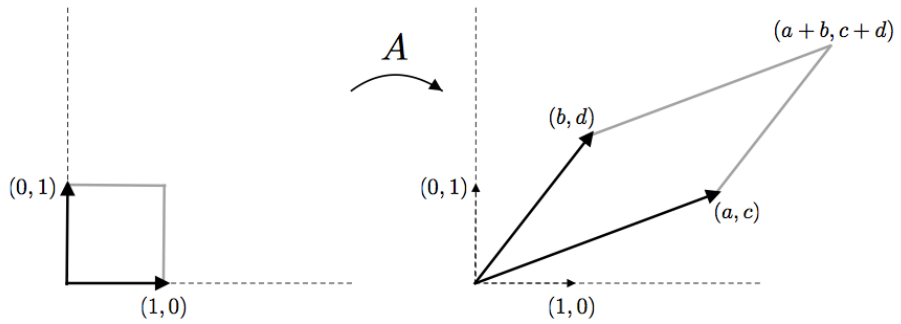


Figure 2.: How the matrix A transforms the integer lattice in $\mathbb{R}^2$.

necessary structures of the commutative torus is given and a summary of the fuzzy torus is given, with an emphasis on the features that will arise in the subsequent chapters.

The flat torus is described mathematically by $\mathbb{R}^2/(2\pi\mathbb{Z})^2$. The standard metric is obtained by the identifying the opposite sides of a square with side length 2π. This will be referred to as the unit square torus. Other metrics are obtained by pulling back the flat metric δ by a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{104}$$

viewed as a integer lattice transformation, where $a, b, c, d \in \mathbb{Z}$ and A is invertible. The new metric is of the form (ignoring the factor of 2π):

$$ds^2 = (a^2 + c^2)d\theta^2 + 2(ab + cd)d\theta d\phi + (b^2 + d^2)d\phi^2 \tag{105}$$

Thus we have a torus as identifying a parallelogram with principal axes $(a, c), (b, d)$ equipped with the metric above. We have that the scalar Laplacian for this metric is

given by

$$\Delta_{\mathbb{T}^2} = \frac{-1}{(ad-bc)^2}(b^2+d^2)\partial_\theta^2 - 2(ad+cd)\partial_\theta\partial_\phi + (a^2+c^2)\partial_\phi^2 \tag{106}$$

and has eigenvalues

$$\lambda^{\Delta}_{k,l} = \frac{1}{(ad-bc)^2}\left((dk-cl)^2 + (bk-al)^2\right) \tag{107}$$

where $(k,l) \in \mathbb{Z}^2$.

The 2-torus has multiple spin structures, making it markedly different from the sphere, which has a unique spin structure due to it being simply connected. The spin structures on the torus are labelled by the first cohomology group with integer mod 2 coefficients. For the torus, this is $H^1(M,\mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, giving four in total, thus we can refer to the different spin structures by $\Sigma \in \mathbb{Z}_2 \times \mathbb{Z}_2$. The choice of spin structure informs us whether the spinors are periodic or anti-periodic along the two axis of the torus. The choice of spin structure of the torus affects the spectrum of the Dirac operator defined upon it [99], but not the global formula for the Dirac operator.

The Dirac operator has the form:

$$\not{D}_{\mathbb{T}^2} = \frac{1}{ad-bc}\left(\gamma^1(d\partial_\theta - c\partial_\phi) + \gamma^2(-b\partial_\theta + a\partial_\phi)\right) \tag{108}$$

for a $(0,2)$ Clifford module. The eigenvalues of the Dirac operator are of the form:

$$\lambda_{k,l,\pm} = \pm\frac{1}{ad-bc}\sqrt{(dk-cl)^2 + (bk-al)^2} \tag{109}$$

where $(k,l) \in (\mathbb{Z}^2 + \Sigma/2)$. Each eigenvalue appears once for two dimensional spinors[13]. For the spin structure $\Sigma = (0,0)$ we have a zero eigenvalue with multiplicity two.

13 It will be useful to consider 4-dimensional spinors when examining the fuzzy torus.

The algebra for a noncommutative torus is the completion of a free, unital, associative *-algebra, $\mathcal{A}$, generated by two unitary operators, U, V, on a Hilbert space $\mathfrak{h}$, modulo the relation $UV = qVU$ for some complex number q. The case when $q = 1$ is realised as the ordinary torus by setting U, V to be the plane waves, i.e. $U = e^{i\theta}$ and $V = e^{i\phi}$. When the algebra $\mathcal{A}$ is finite dimensional, i.e. U, V are matrices, we refer to this as a finite noncommutative torus. Taking the Hilbert space, $\mathfrak{h}$ to be $M_N(\mathbb{C})$ (as opposed to a direct sum of matrix algebras) we say we are dealing with the an irreducible finite noncommutative torus. One consequence of U, V being finite dimensional matrices is that $q^N = 1$ for some positive integer N, due to U, V only having a finite number of eigenvalues.

A key example of such a choice of U, V are the clock and shift matrices

$$C = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & q & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & q^{N-1} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 0 \end{pmatrix} \tag{110}$$

with the choice of q being a N-th root of unity, i.e. $q = \exp(2\pi i/N)$.

We can now define the fuzzy torus:

Definition 31. The fuzzy torus is defined as follows: Let $U, V, \mathfrak{h}$ be a finite noncommutative torus, let $\mathcal{A} = \langle U, V \rangle / (UV - qVU)$ denote the algebra, let $X, Y \in \mathcal{A}$ such that $XY = QYX$ for some complex number Q, such that[14] $Q^N = 1$. Note that $Q^* = Q^{-1}$. Take a $(0, 4)$ Clifford module and the Dirac operator:

$$D_{X,Y} = \frac{1}{Q^{1/4} - Q^{-1/4}} \sum_i \gamma^i \otimes [K_i, \cdot] + \frac{1}{Q^{1/4} + Q^{-1/4}} \sum_{i<j<k} \gamma^i \gamma^j \gamma^k \otimes \{K_{ijk}, \cdot\} \tag{111}$$

14 For instance, $Q = q^a$ for some integer a.

with

$$K_1 = K_{234} = -\frac{1}{4}(X + X^*), \qquad K_2 = -K_{134} = -\frac{i}{4}(X^* - X), \tag{112}$$

$$K_3 = -K_{124} = \frac{1}{4}(Y + Y*), \qquad K_4 = K_{123} = \frac{i}{4}(Y^* - Y) \tag{113}$$

Note that the K_i are all Hermitian, but the pre-factor of the commutator terms is anti-Hermitian. So this formula is an example of equation (79).

There are number of parameters to choose in the definition of a fuzzy torus. For the purposes of this thesis we will simplify to the following cases. Firstly we will always take $U = C$ and $V = S$ unless otherwise stated and so $q = \exp(2\pi i/N)$ and $\mathcal{A} = \mathbb{C}\langle C, S\rangle \cong M_N(\mathbb{C})$ for a positive integer N. We will also take $\mathcal{H} = M_N(\mathbb{C})$. The choice of X, Y plays a pivotal role in the geometry of the fuzzy torus. The simplest case is when $X = C$ and $Y = S$ and so $Q = q$. This corresponds to the fuzzy unit square torus under the outlined in [48].

A result of [48] is that we are not free to specify the spin structure of a fuzzy torus. It is predetermined by the choice of X and Y. It is shown in [48] that the choice of X and Y is equivalent to choosing a matrix A as in (104). If we define $E^{(m,n)} = q^{-mn/2}U^mV^n$. Then by choosing $X = E^{(a,b)} = q^{-ab/2}U^aV^b$ and $Y = E^{(c,d)} = q^{-cd/2}U^cV^d$ $=$, which gives $Q = q^{ad-bc}$, the spin structure of the associated fuzzy torus is then given by the handy formula of $\Sigma = (a + c, b + d)\ mod\ 2$.

The eigenvalues of the associated Dirac operator with $X = E^{(a,b)}$ and $Y = E^{(c,d)}$ are of the form:

$$\lambda_{k,l,\pm} = \pm\frac{1}{[ad - bc]_q}\sqrt{[al - bk]_q^2 + [dk - ck]_q^2} \tag{114}$$

$$= \pm\sqrt{\left[\frac{al - bk}{ad - bc}\right]_Q^2 + \left[\frac{dk - cl}{ad - bc}\right]_Q^2} \tag{115}$$

where $(k, l) \in (\mathbb{Z}_N^2 + \Sigma/2)$ and $[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$ which for $q = e^{2\pi i/N}$ we have $[n]_q = \frac{\sin(\pi n/N)}{\sin(\pi/N)}$ are known as *quantum integers*. We can uniquely index the eigenvalues

with $-N/2 \leq k, l < N/2$. The quantum integers have the property that $[n]_q \to n$ as $q^{1/2} \to 1$ [15].

A key feature to note is that we have doubled the Clifford module from a $(0,2)$ type in the continuum case to a $(0,4)$ type for the fuzzy torus. This has the effect that each eigenvalue (both the positive and the negative square roots) occur with multiplicity two. This is likely related to the Nielsen-Ninomiya theorem [100, 101], which states that a lattice theory of chiral fermions necessarily requires a doubling of the fermions. Which suggests that using this fuzzy geometry does not circumvent this no-go theorem.

There is a subtlety that the value of N affects the spectral geometry quite substantially. If N is divisible by $ad - bc$, then we have that $Q = q^{ad-bc}$ has order $N/(ad-bc)$, i.e. $Q^{N/(ad-bc)} = 1$. So the formula for the quantum integers

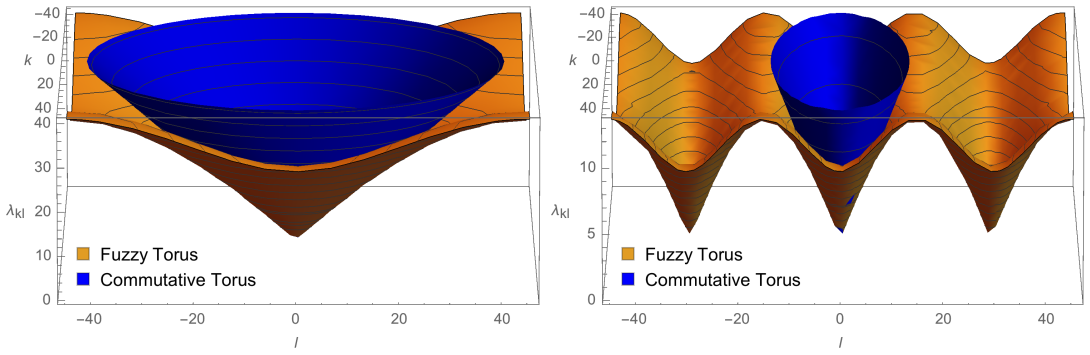
$$[n]_Q = \frac{\sin\left(n\pi\frac{ad-bc}{N}\right)}{\sin\left(\pi\frac{ad-bc}{N}\right)} \tag{116}$$

repeats itself with period $N/(ad-bc)$. As q has period of N we get that the eigenvalues repeat themselves $ad - bc$ times as k, l are varied. We can interpret this (spectrally) as having $ad - bc$ copies of an $N/(ad-bc)$ irreducible fuzzy $ad - bc$ torus. Note that when N is not divisible by $ad - bc$ then the behaviour is more complicated. This case is considered briefly in [48] and is not studied here.

In this thesis we will focus on two fuzzy tori, the unit square torus and the torus with $a = 3$, $d = 1$ and $b = c = 0$, which will be referred to as the $a = 3$ torus.

In Figure 3 a) we see the positive eigenvalues for the $N = 90$ fuzzy unit square tori shown in orange, with the continuum values shown in blue. The surface shown is interpolated between the eigenvalues and the contour lines are for constant height, i.e. constant eigenvalue. We can see that the eigenvalues for the fuzzy unit square torus agree for a small region around $(k, l) = (0, 0)$ and then start to follow a more sinusoidal path as k, l get larger. As N increases we recover the spectrum of the

15 You can also take the limit where $q^{1/2} \to -1$ but that is not considered here



(a) Unit square torus

(b) $a = 3$ torus

Figure 3.: Plots of the positive Dirac spectrum for $N = 90$ fuzzy tori vs their continuum counterparts. b) shows the duplication of the spectrum due to the non-square geometry of the fuzzy torus and $N = 90$ being divisible by 3.

commutative torus in a point-wise manner but with twice the multiplicity. Meaning that for each fixed (k, l) the fuzzy eigenvalue $\lambda_{k,l,\pm}$ converges to the continuum value. However note that at any finite N, the vast majority of the fuzzy spectrum is very different to the continuum values.

Figure 3 b) shows the positive eigenvalues for the $N = 90$ fuzzy and continuum $a = 3$ tori. Here we see an important feature of non-square fuzzy tori. The fuzzy torus is reducible, in that the spectrum gets duplicated, as $N = 90$ is divisible by $ad - bc = 3$. Also the maximum magnitude eigenvalue attained is similar to that of the $N = 30$ fuzzy unit square torus. This will have an effect on some of the spectral measurements we investigate in the next chapter.

2.3.3 *Random Fuzzy Spaces*

The formula's for the fuzzy Dirac operators given by equations (79) and (80) opened up new possibilities. If we fix a Clifford type and a matrix size, the entries of the formula are just Hermitian and anti-Hermitian matrices with no further constraints[16].

16 In [61] they consider L_i matrices that are traceless due to the fact that the trace vanishes in the expression of the Dirac operator as they appear in commutators.

This allows us the possibility of randomising those entries, forming a path integral over the space of geometries

$$Z(\beta, S) = \int_{\mathcal{G}} e^{-\beta S(D)} \mathcal{D}D. \tag{117}$$

This setting lends itself to being studied by Monte Carlo methods, as was done in [61]. A brief account of the procedure is recalled below along with their findings.

This process is then a random matrix model (which are well studied) but where the matrices are restricted to be Dirac operators in the finite spectral triple sense. If we set $\mathcal{G}$ to be the space of Dirac operators, this is a $\mathbb{R}$-vector space as shown in the previous section. We can then take a probability distribution such that a Dirac operator D_j occurs with probability:

$$P(D_j) = \frac{e^{-S(D)}}{\sum_i e^{-S(D_i)}} \tag{118}$$

for some real-valued action $S(D)$. Thus we need to only modify the action to investigate different models.

Note that due to the Dirac operator being of the form in equation (79), the integral over Dirac operators can be expressed as an integral over the (anti-)Hermitian matrices H_i, L_j, so the integration measure is $\mathcal{D}D \propto \Pi_{ij} \mathcal{D}H_i \mathcal{D}L_j$. Writing the anti-Hermitian matrices $L_j = iH_k$, where H_j is a Hermitian matrix reduces the integration $\mathcal{D}D$ to integrating over Hermitian matrices. It is well known in random matrix theory that the integration measure for Hermitian matrices can be written as integrating over arbitrary matrix elements with the Jacobian contributing a factor $\Pi_{i<j} |\lambda_i - \lambda_j|^2$, where λ_i are the eigenvalues of D [102]. This term plays an important role as it can be absorbed into the action to form a new action of the form:

$$\widehat{S}(D) = S(D) + \sum_{i<j} \log |\lambda_i - \lambda_j|^2 \tag{119}$$

which provides an eigenvalue repulsion to the system countering the action $S(D)$. This stops the eigenvalues of the system from settling in any wells of the action potential. It also stops the system from producing eigenvalues with high multiplicities. Recall that the Barrett fuzzy sphere has eigenvalues $j \in \pm\{1, 2, 3, \ldots, N-1\}$ with multiplicities $4|j|$ and $j = \pm N$ with multiplicity $2N$. Thus it is unlikely for an exact fuzzy sphere to arise in from the simulation for the low matrix sizes considered below.

The only actions considered are those that are 'spectral', in the sense that they only depend on the eigenvalues of the Dirac operator:

$$S(D) = TrV(D) = \sum_{\lambda_i \in spec(D)} V(\lambda_i) \tag{120}$$

for some potential V bounded from below. We also need to make sure that $\int_{\mathcal{G}} e^{-S(D)} dD$ converges in order to study it using Monte Carlo methods. This constrains the possible actions we can choose. This restriction excludes the Connes-Chamseddine spectral action [39],

$$S(D, \Omega) = Tr(f(D/\Omega)), \tag{121}$$

where f is a smooth approximation to a step function to regularise the trace and Ω is a energy scale cutoff scale. This action can recover the standard model action and the Einstein-Hilbert action of GR as was discussed in the introduction. However, this action does not have a well localised minima as it tends to zero asymptotically. Thus is not appropriate for use in a random matrix model as the system will not thermalise. In fact, it is needed that the potential $V(x) \to \infty$ as $x \to \infty$ in order for the minimum to be sufficiently well localised.

In [61] they studied the simplest actions satisfying this constraint, which take the form

$$S(D) = g_2 Tr(D^2) + g_4 Tr(D^4), \tag{122}$$

with $g_4 > 0$ or $g_4 = 0$ and $g_2 > 0$. As the space of Dirac operators $\mathcal{G}$ is a real vector space, we can scale the Dirac operator D so that we can assume $g_4 = 1$ or

$g_4 = 0$ and $g_2 = 1$. When $g_4 = 1$ and $g_2 < 0$, the potential forms a double well, which is known to lead to a phase transition in random matrix models. This was also found in [61] and investigated further in [63] for a number of different Clifford types. When $g_4 = 0$ and $g_2 = 1$, no phase transition was found and this model will not be analysed here.

Monte Carlo Procedure

Once an action has been chosen, the matrix model was then setup to undergo Markov Chain[17] Monte Carlo simulation using the Metropolis-Hastings algorithm. In which a new Dirac operator D_{j+1} is generated from D_j by generating random matrices, R_i, one for each randomised entry into the Dirac operator. We then define $\delta H_i = \mathfrak{l}(R_i + R_i^*)$ and $\delta L_i = \mathfrak{l}(R_i - R_i^*)$, where $\mathfrak{l}$ is real valued and determines the step length in the configuration space. The parameter l undergoes adjustment to make the Monte Carlo algorithm thermalise. We then define $D_p = D_j + \delta D$, where δD is constructed linearly from the $\delta H_i, \delta L_i$. This new proposed Dirac operator D_p is accepted (i.e. $D_{j+1} = D_p$), if one of the following conditions hold:

- $S(D_p) < S(D_j)$
- $\exp\big(S(D_j) - S(D_p)\big) > \mathfrak{p}$ where $\mathfrak{p}$ is a random number chosen uniformly from $[0, 1]$

If the proposed Dirac operator D_p fails both conditions we just set $D_j = D_{j+1}$. After a sufficient number of moves we will have that the steps are independent of the choice of initial operator D_0.

Results of previous studies

The random fuzzy spaces investigated in [63] have Clifford-type $(1, 1)$, $(2, 0)$ and $(1, 3)$. All were found to undergo a phase transition when using the action $S =$

17 This means that the generated Dirac operators, D_{j+1} only depend on the current Dirac operator D_j and not D_{j-1}, D_{j-2}...

$g_4 Tr(D^4) + g_2 Tr(D^2)$ upon varying g_2, the location of the phase transition was different for each type however. It is these random geometries that will be investigated in this work also. The location of the phase transition is found by examining some thermodynamic variables under the partition function $Z(\beta, S)$. For the action above we have two variables available, the 'inverse temperature' β and the coupling constant g_2 (as $g_4 = 1$). Thus we have the following thermodynamical quantities to investigate:

$$\langle S \rangle = -\frac{\partial \log(Z)}{\partial \beta} \qquad Var(S) = \frac{\partial^2 \log(Z)}{\partial \beta^2} \tag{123}$$

$$\beta \left\langle Tr(D^2) \right\rangle = -\frac{\partial \log(Z)}{\partial g_2} \qquad \beta^2 Var(Tr(D^2)) = \frac{\partial^2 \log(Z)}{\partial g_2^2} \tag{124}$$

For a infinite system, the variance of both the action and the variance of $Tr(D^2)$ should diverge at a phase transition if it is second order or higher, and for a finite size system they should exhibit a pronounced peak. This was clearly found for the type $(2,0)$ and $(1,3)$ geometries, with $Var(S)$ shown in Figure 4 of [63] and $Var(Tr(D^2))$ shown in Figure 8 of [63]. However, for the type $(1,1)$ geometry, both plots show a qualitative difference to the other two types indicating that it might have a higher order phase transition or that larger matrix sizes are need to further investigate.

For each type and matrix size the location of the phase transition can be found in [63] and the calculated averages are given in table 3. The ranges of g_2 values investigated was different for each Clifford type, for type $(1,1)$ g_2 was taken between $[-3.50, -2.00]$, for type $(2,0)$ between $[-3.50, -2.50]$ and for type $(1,3)$ between $[-4.00, -3.30]$ starting at the least value in the respective ranges and increasing with a step size of 0.05. We call g_2 values that are more negative as being after the phase transition and the g_2 value that are less negative as being before the phase transition.

The scaling freedom within the action studied here, has the effect that the maximum eigenvalue of the geometries is restricted. Thus as we increase the matrix

Table 3.: Table of the phase transition value of g_2 for the three Clifford types of random geometry investigated. The average is calculated using the data from [63].

Type	$(1,1)$	$(2,0)$	$(1,3)$
Critical g_2 value	-2.382 ± 0.289	-2.781 ± 0.289	-3.696 ± 0.354

size in the simulations we do not attain larger eigenvalues but more densely pack a finite region. As some of the spectral measures developed in chapter 3 are based upon the asymptotic behaviour of the eigenvalues, this restriction of the eigenvalues may seem problematic. However, as all of the eigenvalues are scaled, this effect can be countered by normalising the spectrum by some overall factor. There are a number of ways to do this, for instance when comparing two geometries we could force the eigenvalues with minimum or maximum magnitude to agree. We could also scale the spectrum by a factor depending only on the matrix size, N, to counter the scaling done in the action. As was discussed in [64] the minimum eigenvalues of the random geometries turns out to be too 'random' for practical use. And scaling to match the maximum eigenvalues and the N-dependent scaling produce similar results. When comparing the random geometries to the other well known fuzzy spaces described above, we scale the random spectrum so the maximum eigenvalue agree with the maximum eigenvalue of the well known geometries. The effect of such a scaling on the geometry is to be determined, but it is expected to be related to changing the 'size' of the geometry[18].

Observables for random fuzzy spaces

An observable is any real or complex valued function of the Dirac operators. The expectation value of an observable f is calculated by performing the integral

$$\langle f \rangle = \frac{1}{Z} \int f(D) e^{-S(D)} \mathcal{D}D. \tag{125}$$

18 For instance, the sphere of unit radius has a Dirac spectrum of $\mathbb{Z}\backslash 0$. The sphere of radius $r \in \mathbb{R}$ has a Dirac spectrum of $\frac{1}{r}$ times the unit sphere spectrum.

This integral can be approximated by taking a discrete ensemble of Dirac operators $\{D_1, D_2, \ldots, D_N\}$ by the formula:

$$\langle f \rangle_N = \frac{\sum_i f(D_i) e^{-S(D_i)}}{\sum_j e^{-S(D_j)}}, \tag{126}$$

which converges to the continuum expression as $N \to \infty$. If the ensemble $\{D_j\}$ is taken from a Monte Carlo simulation of the path integral, the following simplification exists [103]:

$$\langle f \rangle_N = \frac{1}{N} \sum_i f(D_i). \tag{127}$$

In the following chapters, measurements of fuzzy spaces are developed that depend on the eigenvalues of their Dirac operators. Therefore in order to investigate these measurements for the random geometries, their expectation values can be calculated via eq. (127). This method with be referred to as taking the ensemble average.

However as the eigenvalues of the Dirac operators D_j are also observables. By ordering the eigenvalues of each D_j and letting $f^i(D)$ denote the function that selects the ith eigenvalue of D, the expectation value $\langle f^i \rangle$ can be calculated. Thus for a random fuzzy matrix model, with Dirac operators that are N by N matrices, this would result in N averaged eigenvalues of the Dirac operators, denoted by $\{\langle f^i \rangle\}_{i=1}^N$. This collection of eigenvalues $\{\langle f^i \rangle\}$ will be referred to as an *averaged geometry* of the model. This allows us to calculate the spectral measures for these averaged geometries. It should be noted, that averaging the eigenvalues in this way is not standard. However it provides to be a powerful tool in the investigation of the geometry of these random fuzzy spaces. Thus, there are two possible ways to apply the spectral measures discussed in the next chapter. Either by ensemble averaging the quantities by eq. (127) or by calculating the measures for the average geometry of the model.

There is also a third option, which is to collate all of the eigenvalues for each Dirac operator D_j in the ensemble, into one large spectrum and then calculate the measure on this set of eigenvalues. However this method will remove an important feature of

the spectra, the spacing of the eigenvalues and in the limiting case would produce a continuum of eigenvalues. This is undesirable as none of the individual geometries having a continuum spectrum. This method is not investigated here.

Using the averaged spectrum $\{\langle f^i \rangle\}$ of all the geometries involved is computationally efficient, however it raises some questions about the meaning of an averaged geometry. It also ignores that the fluctuations of the eigenvalues might be correlated with each other, and this could have important consequences for the spectral estimators. Calculating the ensemble average of the dimensional/volume measures is therefore the preferred method, however it is computationally taxing. An investigation of how constant the spectral estimators are over the ensemble is conducted and used to justify whether using the averaged spectrum is a legitimate method to study these models. An explicit comparison between the spectral measures of the averaged spectrum and the ensemble average of the spectral measures is considered below. A priori these methods are not required to produce similar results.

3

SPECTRAL MEASURES OF FUZZY SPACES

In the commutative geometry setting, there are many different definitions of the dimension of a space that are equivalent. For smooth manifolds, the simplest is the minimum number of coordinates needed to specify any point within it. This is equivalent to the minimum number of independent basis elements in each tangent space. With the discovery of fractal spaces, new notions of dimension were defined which need not take integer values [104]. In non-commutative geometry, the dimension is usually replaced with the *dimension spectrum*, which is no longer a single number, but a subset of the complex plane. It is defined as the set of singularities of the zeta function of the Dirac operator [105]. The K0 dimension is another notion of dimension that for compact spin manifolds, is equal to the topological dimension modulo 8. See [106] for an overview of the various dimensions available in geometry and algebra. Many of these notions generalise to the noncommutative geometry setting, however they no longer necessarily agree. Also, these definitions tend to be trivial when applied to finite spectral triples. However, we have seen that we have finite non-commutative approximations to the sphere and the torus and as more examples are uncovered, a notion of dimension that captures this feature is desirable.

The notion of the volume of a space is more narrowly defined, with the definition stemming from prescribing a volume form, i.e. a differential form of top degree. There are a number of ways to compute the volume depending on the context. In terms of spectral geometry we will see it can be computed by probing the asymptotic expansion coefficients of the heat kernel of a differential operator [107]. It can be

also calculated by a *noncommutative integral* over a generalised notion of a volume form [65]. This notion of a volume form is given in the more abstract terms of a Hochschild cocycle, which also encodes an orientation. However, the details of this are not needed as the expression for the noncommutative integral can be given in terms of a singular trace is outlined below.

In what follows we investigate a number of different approaches to calculate the dimension and volume of a fuzzy space, using just the spectrum of their Dirac operators. The results of this section were published in [64].

3.1 SPECTRAL GEOMETRY

One geometric pursuit that is perfectly generalisable to noncommutative setting is that of spectral geometry. This is the study of a geometrical space by examining the eigenvalues of operators defined upon it. The archetypical example is by studying the Laplace operator for which a number of powerful statements exist, with Weyl's law being the most celebrated.

Theorem 3 (Weyl's Law [108])**.** *Given a compact oriented Riemannian manifold, M, and the Laplace-Beltrami operator. The number $N(\lambda)$ of eigenvalues less than λ satisfies*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{Vol(M)}{(4\pi)^{d/2}\Gamma(d/2+1)} \tag{128}$$

If we order the eigenvalues so that $\lambda_0 \leq \lambda_1 \leq \dots$ we can restate this theorem as an asymptoptic approximation for the k-th eigenvalue:

$$\lambda_k \sim 4\pi \left(\frac{Vol(M)}{\Gamma(d/2+1)}\right) k^{2/d} \tag{129}$$

It was shown that knowledge of the Laplace spectrum alone is not enough to characterise a space. This was done by the creation of isospectral but not isometric versions of a 16-dimensional torus by Milnor [109]. These are different (non-isometric) Riemannian manifolds (M, g) and (M', g') such that $spec(\Delta) = spec(\Delta')$ and an-

swered the adage of 'Can you hear the shape of a drum?' with a resounding no for the spectrum of the scalar Laplacian [110]. Isospectral deformations of Riemannian Spin manifolds exist also [111], meaning that knowledge of the spectrum of the Dirac operator is not enough to specify a Riemannian manifold (M, g).

However, it was shown that if you know the spectrum of a differential operator, and the relative position of two commutative von Neumann algebras within the Hilbert space on which the differential operator acts, then this uniquely determines a Riemannian manifold [112]. This holds for the Laplace-Beltrami operator and also for the Dirac operator acting on a spinor bundle. Note that if we take the Dirac operator on a spin manifold, we can not recover the precise Dirac operator back from the spectrum, as we only fix it's principal symbol i.e. we get a Dirac-type operator not *the* Dirac operator. We do however fix the spin structure if we know the volume form (or its cohomological content) [113, section 11].

Regardless of the existence of isospectral Riemannian manifolds, the formula (129) clearly shows that the spectrum of the Laplacian contains information about the dimension and volume of the manifold. Thus is it possible to extract this information from the spectrum. Powerful methods of extracting geometrical data from the spectrum of differential operators have been developed and will be outlined below. These are mostly centred around analysing the heat kernel asymptotic expansion. As the notion of the spectrum of an operator survives when we pass to noncommutative geometry, one can hope to find a similar characterisation of the 'dimension' and 'volume' of a noncommutative geometry this way.

3.1.1 *Weyl's Law for Fuzzy Spaces*

Weyl's law for spectral triples requires us to deal with the Dirac operator and spin bundles, whereas the original result holds for only the Laplace-Beltrami operator over a Riemannian manifolds. However similar results hold for operators of *Laplace-type*

Definition 32. A differential operator, $L = -(a^{\mu\nu}\partial_\mu\partial_\nu + a^\alpha\partial_\alpha + a^0)$, on smooth sections of a vector bundle, $V \xrightarrow{\pi} M$, over a Riemannian manifold, M, where $a^{\mu\nu}, a^\alpha, a^0 \in End(V)$, is said to be of Laplace type if $a^{\mu\nu} = g^{\mu\nu}id$. A differential operator D is said to be of Dirac type if D^2 is of Laplace type. If we expand $D = \gamma^\mu\partial_\mu + \gamma_0$, then D is of Dirac type if and only if we have that $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}id$.

Remark 4. *We want the Laplace-Beltrami operator to have positive eigenvalues. Under this definition we have that the Laplacian on flat space is given by $\Delta = -g^{\mu\nu}\partial_\mu\partial_\nu$, so Δ has positive eigenvalues. Other authors use the convention that $\Delta = g^{\mu\nu}\partial_\mu\partial_\nu$, in which $-\Delta$ would have positive eigenvalues.*

Theorem 4 ([108])**.** *Given a compact Riemannian spin manifold, M with a spinor bundle of rank k and the Dirac operator with eigenvalues λ_i where $\{|\lambda_i|\}$ is non-decreasing, we have that*

$$\lim_{n\to\infty} \frac{n}{|\lambda_n|^d} = \frac{kVol(M)}{(4\pi)^{d/2}\Gamma(1+\frac{d}{2})} \tag{130}$$

where n indexes the eigenvalues.

Note that the right hand side of this expression is constant for a given geometry and the sequence, $\{n/|\lambda_n|^d\}$ takes values in the positive reals. As the logarithm is a continuous function on the positive reals we have that

$$\log(n) - d\log(|\lambda_n|) \to const = C = \log\left(\frac{kVol(M)}{(4\pi)^{d/2}\Gamma(1+\frac{d}{2})}\right) \tag{131}$$

The equation eq. (131) provides us with a possible procedure to calculate the dimension and volume from the spectrum. Taking the approximation that

$$\log(n) = d\log(|\lambda_n|) + C, \tag{132}$$

$$Vol(M) = \frac{1}{k}(4\pi)^{d/2}\Gamma(1+\frac{d}{2})\exp(C) \tag{133}$$

for every n. The dimension d and the volume could then be calculated by fitting a straight line to the graph of $\log(n)$ vs $\log(|\lambda_n|)$. However, as the expression eq. (131)

only holds asymptotically, a naive attempt to fit a straight line to the plot of $\log(n)$ vs $\log(|\lambda_n|)$ is inappropriate. In order for this procedure to produce reasonable estimations for the dimension and volume, the linear regression must be fitted to only the largest eigenvalues. However, how to determine when the eigenvalues are 'large enough' to get accurate results is still a question one must answer.

The situation is worse for the fuzzy spaces that this thesis is concerned with. As Weyl's law is essentially a "high energy" (i.e. large eigenvalue) investigation into the structure of the geometry. If you consider the Dirac spectrum for the unit square fuzzy torus, the eigenvalues deviate drastically from the continuum spectrum for their largest values. This was shown in fig. 3, where it can be seen that the eigenvalues of the fuzzy torus and continuum agree for only a small region around the origin. Thus if a procedure based upon Weyl's law is developed to analyse the largest eigenvalues only, this would not be effective for the fuzzy spaces in general. As the highest eigenvalues do not typically obey the same behaviour as their continuum counterparts. We can view this as the finite/noncommutative effects of the fuzzy spaces having the effect of altering the manifold-like geometry for the highest energies. For some spaces such as the fuzzy sphere, these features are minimal, as the spectrum is just a truncation of the continuum spectrum. For other spaces like the fuzzy tori and possibly others, these quantum effects are very prevalent.

Thus we don't necessarily want to have a definition of dimension based on the highest eigenvalues present. We also do not want a dimension measure based on the smallest eigenvalues as they encode some of the global geometry which will also skew the dimensional measures [69, Chapter 5]. So we would need to use the growth behaviour of the some middling region of the spectrum at each value of n. How to choose such a region is not obvious and is not currently understood. To demonstrate this point we examine another manipulation of Weyl's law.

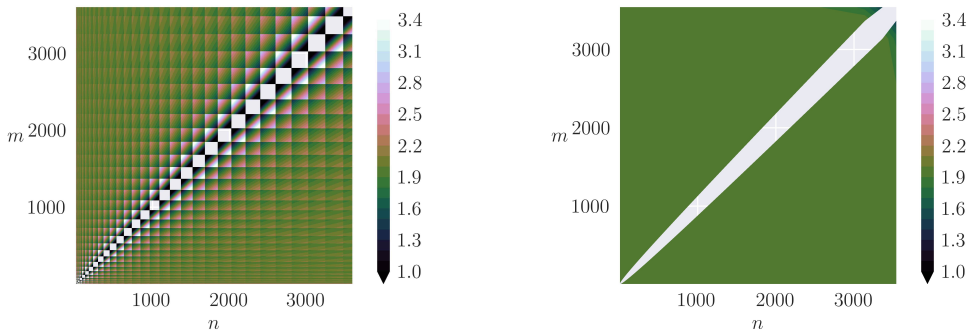


Figure 4.: A two parameter plot of W_{nm} for the fuzzy sphere of matrix size $N = 30$. First plot shows the multiplicity blocks for each eigenvalue, the second plot is corrected for this.

Another method to extract the dimension from Weyl's law is to take two different (distinct) eigenvalues, λ_n, λ_m and examine the difference of eq. (131) for each other:

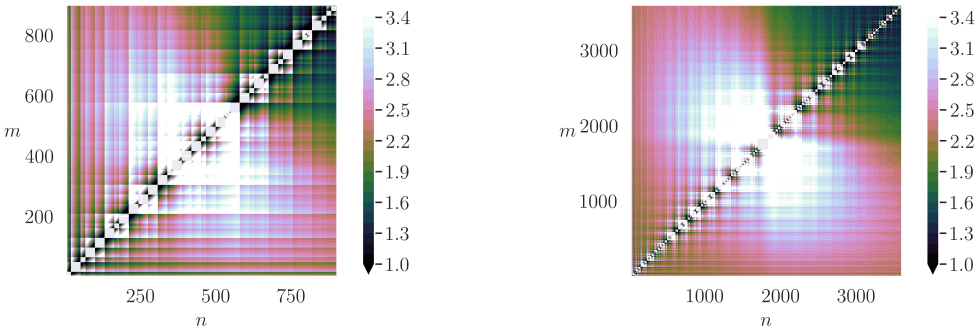
$$\log(n) - d\log(|\lambda_n|) - (\log(m) - d\log(|\lambda_m|)) = \log(n/m) - d\log(|\lambda_n|/|\lambda_m|) \to 0. \tag{134}$$

The limit is to taken so that $\log(n/m)$ is bounded and we require that $\lambda_n \neq \lambda_m$. In such a case we can define the following expression:

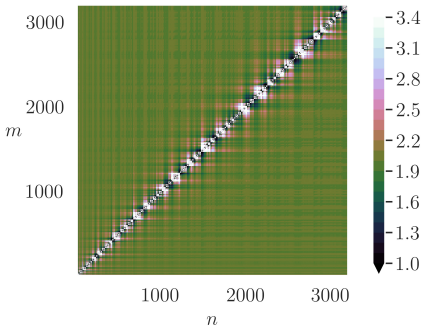
$$W_{nm} = \frac{\log(n/m)}{\log(|\lambda_n|/|\lambda_m|)} \to d \tag{135}$$

This expression is invariant under rescaling of the eigenvalues, which will be useful when investigating the random geometries, as they are arbitrarily scaled in the simulation to restrict the maximum eigenvalue, so only the density of the eigenvalues varies as n increases.

We have plotted the function W_{nm} for the fuzzy sphere in fig. 4. The white squares along the diagonal are where the two eigenvalues λ_n and λ_m are equal and thus W_{nm} is ill-defined. The large multiplicities of each eigenvalue cause the value of W_{nm} to vary as we examine each multiplicity block. This can be seen in the left-hand plot where it causes a psychedelic pattern, whereas in the right-hand plot the value for the middle of each multiplicity block is shown. This second plot shows that the



(a) $N = 15$ (b) $N = 30$

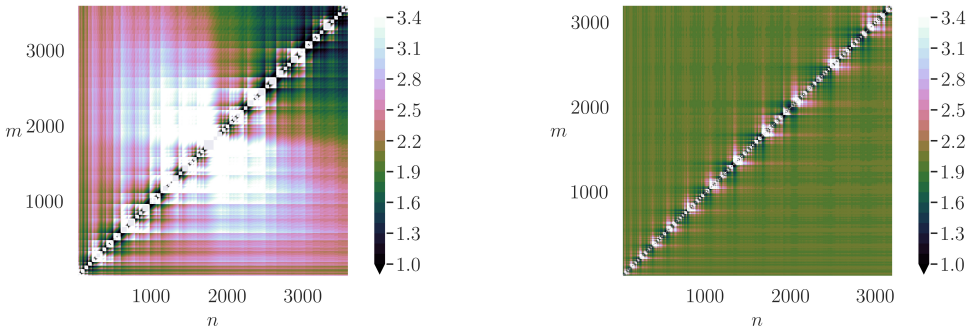


(c) Continuum torus

Figure 5.: A two parameter plot of W_{nm} for the unit square fuzzy torus of matrix sizes $N = 15$ and $N = 30$, and the continuum unit square torus with the same number of eigenvalues as the fuzzy torus.

fuzzy sphere has a stable notion of dimension throughout it's spectrum, and that value is close to 2.

The situation is drastically different when we consider the fuzzy torus. Figure 5 shows the plot of W_{nm} for the $N = 15$ and $N = 30$ unit square fuzzy torus compared with the continuum counterpart. Here we see that the value of W_{nm} varies wildly depending on where in the spectrum you investigate. The bad behaviour of this function W_{nm} is again due to the large eigenvalues of the fuzzy torus being significantly different their continuum counterparts. This is due to the q-number $[l]_q$ only approximating the integer l for $\pi l/N \ll 1$ but deviates from this as l increases. However their is a small region around the origin in which the function W_{nm} is ap-



(a) $a = 3$ Fuzzy Torus

(b) Continuum $a = 3$ torus

Figure 6.: Plot of W_{nm} for the rectangular Torus with $a = 3$, $d = 1$, $b = c = 0$. Figure a shows the fuzzy torus with matrix size $N = 30$ and b) shows the continuum torus.

proximately 2. There doesn't seem to be a way to characterise how big this region is or its location as we increase the matrix size.

The plot of W_{nm} for the $a = 3$ fuzzy torus is included in fig. 6. Here we see that the despite having the matrix size at $N = 30$, the $a = 3$ fuzzy torus (fig. 6 (a)) appears be to be more similar to that of the lower matrix size $N = 15$ unit torus from 5. The overall characteristics of the plot are not changed much by the change in torus shape.

In fig. 7 the function W_{nm} is plotted for the type $(2, 0)$ geometry with matrix size $n = 10$ and with g_2 values around the phase transition value. The value of W_{nm} again varies depending on where you look in the spectrum, however, it is much better behaved than the fuzzy torus example for certain values of g_2. W_{nm} takes

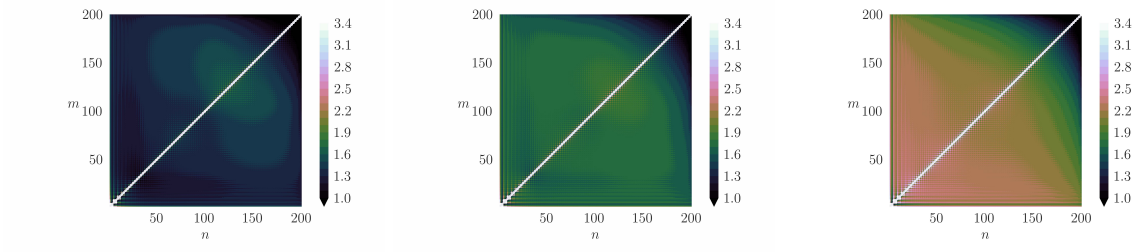


Figure 7.: A two parameter plot of W_{nm} for the type $(2, 0)$ random geometries, with g_2 taking values of -2.75, -2.80 (phase transition) and -2.85.

values between $1.2 - 1.6$ at $g_2 = -2.75$ and between $1.6 - 2.0$ at $g_2 = -2.80$. After the phase transition their is no stable value, demonstrated in the plot for $g_2 = -2.85$.

An interesting feature is that the value of W_{nm} drops to zero at the largest eigenvalues. This feature was present in the fuzzy torus plots (fig. 5 and fig. 6) as well as for the random geometries. It is interpreted as the discrete nature of the fuzzy spaces being encoded. Weyl's law is the most naive spectral measurement of a geometry, more advanced methods exist and are explored in the following section.

3.2 HEAT KERNELS, SPECTRAL ZETA FUNCTIONS AND GEOMETRY

The heat equation is given usually in terms of the Laplace operator[1] in the following way: Let M be a compact Riemannian manifold with Laplace-Beltrami operator Δ, then the solution to the following equation are called the heat kernel for the operator Δ:

$$(\frac{\partial}{\partial t} + \Delta_x)K(x, y; t) = 0 \tag{136}$$

where K is a function on $M \times M \times \mathbb{R}^+$ subject to the initial condition that as $t \to 0$, $K(x, y; t)$ is the Dirac delta distribution along the diagonal $x = y$. It is sometimes denoted $K(x, y; t, \Delta)$ to show the dependence on the differential operator when it is not obvious from context. The solution to this equation, K, is smooth and its behaviour as $t \to 0$ encodes the local geometry of the manifold M. We can write down the formal solution as:

$$K(x, y; t, \Delta) = e^{-t\Delta}\delta(x - y) \tag{137}$$

where $e^{-t\Delta}$ is given by the functional calculus, i.e. $e^{-t\Delta} = 1 - t\Delta + t^2\Delta^2 + \dots$. As Δ is self-adjoint, we have a spectral resolution, $\{\lambda_n, \phi_n\}$, such that $\Delta\phi_n = \lambda_n\phi_n$ and

1 Depending on your metric signature the heat equation may have a negative sign infront of Δ_x.

$\{\phi_n\}$ form an orthonormal basis on the L^2 sections. In this basis we can express the heat kernel as:

$$K(x, y; t, \Delta) = \sum_n e^{-t\lambda_n} \phi_n(x) \otimes \phi_n^*(y) \tag{138}$$

We can then view the heat kernel as an operator from the fibre above y to the fibre above x. The heat kernel has an associated integral transform, which gives a representation of the operator $e^{-t\Delta}$:

$$(e^{-t\Delta}\phi)(x) = \int_M K(x, y; t, \Delta)\phi(y)dy. \tag{139}$$

We then consider the case when $x \to y$, which is encoded in the L^2 trace over the bundle $V \xrightarrow{\pi} M$.

$$K(t, \Delta) := Tr_{L^2}(e^{-t\Delta}) = \int_M tr_{V_x} K(x, x; t, \Delta)dx = \sum_n \exp(-t\lambda_n) \tag{140}$$

We will often refer to the trace of the heat kernel as simply the heat kernel.

The above heat kernel can be constructed for any elliptic pseudo-differential operator and possess an asymptotic expansion as $t \to 0 \in \mathbb{R}^+$. Focussing on Laplace-type operators, P, we have that [114]:

$$K(t, P) \overset{t\to 0}{\sim} t^{-d/2} \left(a_0(P) + a_1(P)t + a_2(P)t^2 + \dots \right) \tag{141}$$

where d is the dimension of the manifold M and $a_i(P)$ can be expressed in terms of local geometric quantities.

Specifying to a compact Riemannian manifold without boundary, equipped with the Dirac operator squared, D^2, as our Laplace-type operator we have that the odd

indexed coefficients all vanish (shown in [114]) and the first two even coefficients can be expressed as:

$$a_0(D^2) = \frac{1}{(4\pi)^{\frac{d}{2}}} \mathrm{Tr}(Id) \int_M d^d x \sqrt{g} \tag{142}$$

$$a_2(D^2) = -\frac{1}{(4\pi)^{\frac{d}{2}}} \frac{1}{12} \mathrm{Tr}(Id) \int_M d^d x \sqrt{g} R \tag{143}$$

where the Tr is taken over the fibre and $Tr(Id)$ is equal to the rank of the bundle. Higher coefficients have been computed also, but we will not be examining them in this work.

How one computes each of these coefficients from the spectrum of a differential operator alone is not given to us via this path of thought. However there is a spectral method which selects the coefficients. We do this by making use of the spectral zeta functions, a procedure that clearly carries over to the noncommutative geometry setting as we shall see.

3.2.1 *Spectral Zeta functions and Heat Kernel Coefficients*

Let $\{\lambda_i\}$ be a discrete set of non zero eigenvalues of a Dirac operator, D. Then for sufficiently large $Re(s)$, we define the spectral zeta function of the Dirac operator as[2]:

$$\zeta_{D^2}(s) = \sum_i \left(\lambda_i^2\right)^{-s} . \tag{144}$$

This is an easily computed entity for a finite spectrum and for some simple cases of infinite spectra a closed form can be found in terms of well known functions. However for most spaces, the spectral zeta function is not expressible in such terms.

2 In order for the zeta function to be expressed as a sum, the space is assumed to be compact and without boundary, however there do exists non-compact manifolds with discrete Dirac spectrum (such as some link complements in $\mathbb{R}^3$, see [99]). For the finite spectral triples we aim to explore, the sum is also well defined.

We shall see that it is just the residues of this function at certain poles that are of importance to us.

The spectral zeta function can be shown to be the the Mellin transform of the spectral heat kernel [115]:

$$\zeta_{D^2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t) dt \tag{145}$$

It can be shown that the coefficients in the heat kernel asymptotic expansion, can be related to the residues of the spectral zeta function and Euler's gamma function [116]. We state the result for the square of a Dirac-type operator:

$$a_k = Res_{s=\frac{d-k}{2}}(\Gamma(s)\zeta_{D^2}(s)) \tag{146}$$

Thus considering $k = 0$ in this formula, we see that that a pole in the spectral zeta function is located at precisely $s = \frac{d}{2}$. Thus locating the most positive value of s where the spectral zeta function has a pole provides us with a way to measure the dimension of a space.

Also, by using eqs. (142) and (143) with eq. (146) we can express some familiar geometric quantities in terms of the residues of the spectral zeta function:

$$Vol(M) = \frac{(4\pi)^{d/2}}{\mathrm{Tr}(Id)} Res_{s=\frac{d}{2}}(\Gamma(s)\zeta_{D^2}(s)) \tag{147}$$

$$\int_M d^d x \sqrt{g} R = -12 \frac{(4\pi)^{d/2}}{\mathrm{Tr}(Id)} Res_{s=\frac{d}{2}-1}(\Gamma(s)\zeta_{D^2}(s)). \tag{148}$$

Equation (147) will be examined in detail in section 3.4. In order to make use of eq. (148) to investigate fuzzy spaces, a notion of analytic continuation for the finite spectral zeta function has to be developed. The recent work by Stern [66] appears to provide a methodology for this, but it is not explored here.

Remark 5. *Note that these statements hold for much more general settings. See [117] and Theorem 3.5 for example in [116].*

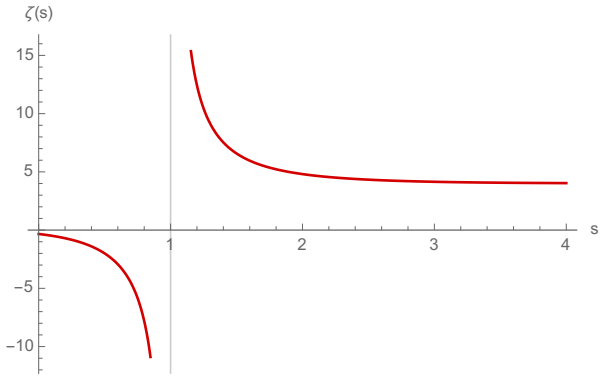


Figure 8.: The spectral zeta function for the 2-sphere, demonstrating that the pole occurs at $s = d/2 = 1$.

We can investigate this procedure for the continuum 2-sphere with relative ease. Given that the Dirac spectrum for the 2-sphere is just the integers excluding zero with the integer j occurring with multiplicity $2|j|$, we have that

$$\zeta_{\not{D}^2_{S^2}}(s) = \sum_{j=1}^{\infty} 2|j|(j^2)^{-s} + 2|j|((-j)^2)^{-s} = \sum_{j=1}^{\infty} 4j^{1-2s} \tag{149}$$

$$= 4\zeta_R(2s-1) \tag{150}$$

where $\zeta_R(s)$ is the Riemann Zeta function, which has a pole at $s = 1$. So for the 2-sphere spectral zeta function we have that $\zeta_{\not{D}^2_{S^2}}(s)$ has a pole at $2s - 1 = 1$, i.e $s = 1$. But as the pole occurs at $s = d/2$ we have that $d = 2$.

For the 2-torus the spectral zeta function gets more complicated due to it's more complicated spectrum. Recall that the flat continuum 2-torus has a Dirac spectrum of

$$\lambda_{k,l,\pm} = \pm\frac{1}{ad-bc}\sqrt{(dk-cl)^2 + (al-bk)^2} \tag{151}$$

with $k, l \in (\mathbb{Z}^2 + \Sigma/2)$ where Σ denotes the spin structure. Each eigenvalue appears once when dealing with two dimensional spinors. Note that the spin structure $(0,0)$ also has a zero eigenvalue with multiplicity two for two dimensional spinors.

Taking any spin structure (as we do not include zero eigenvalues in the spectral zeta function) we have that:

$$\zeta_{\not{D}^2_{\mathbb{T}^2}}(s) = \sum_{k,l\in(\mathbb{Z}^2+\Sigma/2)} \frac{2}{(ad-bc)^{-2s}}((dk-cl)^2+(al-bk)^2)^{-s} \tag{152}$$

simplifying to the square torus $a = d = 1$, $b = c = 0$ with the $(0,0)$ spin structure, we have that

$$\zeta_{\not{D}^2_{\mathbb{T}^2}}(s) = \sum_{\substack{k,l=0 \\ (k,l)\neq(0,0)}}^{\infty} 4(k^2+l^2)^{-s} \tag{153}$$

which is an Epstein zeta function in two dimensions, which is known to converge absolutely for any $Re(s) > 1$ and have a pole located at $s = 1$ (see [118, Section 1.4]).

For the fuzzy spaces in question, the spectral zeta function is a finite sum, and thus has no poles to locate. However, there is a way to investigate the potential location of poles as we increase the matrix size of the fuzzy spaces. To see how one does this, consider a continuum space and recall Weyl's law for the eigenvalues of the Dirac operator. Specifically that $|\lambda_k| \sim k^{1/d}$ where the constants to make this relation an equality depend on the volume of the space. Substituting this expansion into the spectral zeta function at the value of $s = d/2$ we see that

$$\zeta_{D^2}(\frac{d}{2}) \sim \sum_{k=1}^{\infty}((k^{1/d})^2)^{-d/2} = \sum_{k=1}^{\infty} k^{-1} \tag{154}$$

which is well known to logarithmically diverge, i.e.

$$\zeta_{D^2}(d/2) \sim \lim_{n\to\infty} \sum_{k=1}^{n} k^{-1} \sim \lim_{n\to\infty}(\log(n)+\gamma+O(1/n)), \tag{155}$$

where γ is the Euler-Mascheroni constant.

So noticing that for finite spectra,

$$\zeta_{D^2}(0) = \sum_{k=1}^{N} \lambda_k^0 = N \tag{156}$$

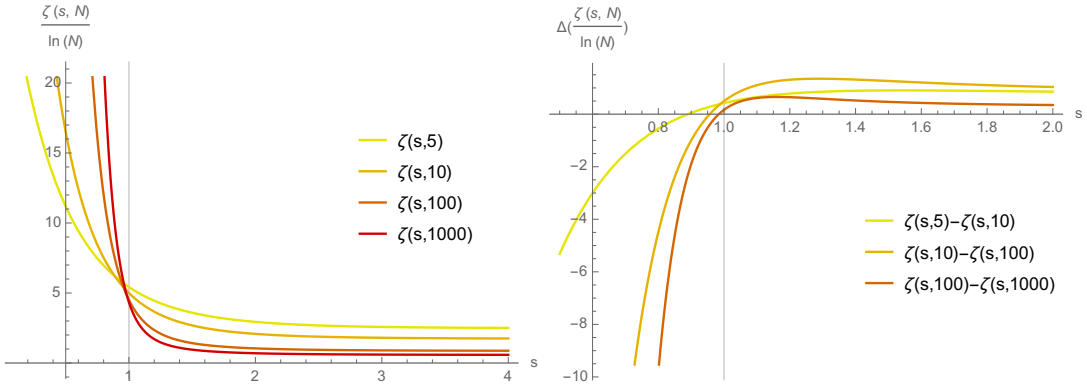


Figure 9.: Logarithmically scaled zeta function for the fuzzy sphere of various sizes showing the location of the pole in the infinite limit.

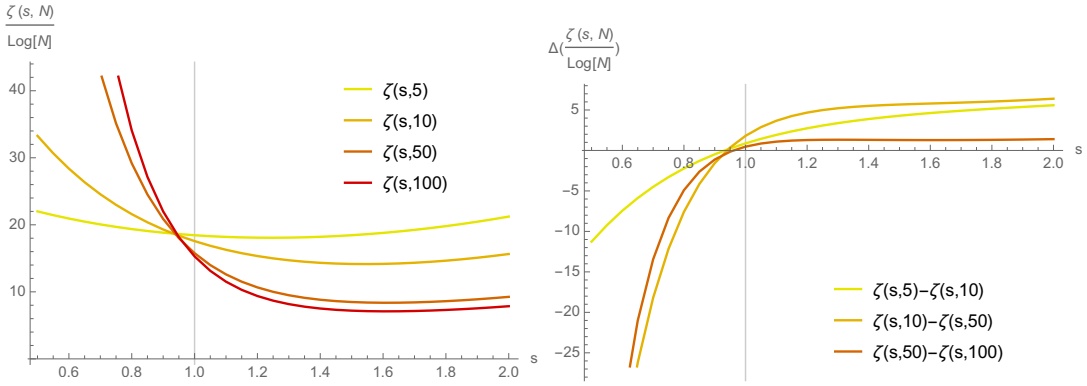


Figure 10.: Logarithmically scaled zeta function for the fuzzy torus of various sizes.

counts the number of eigenvalues. If we examine the following function $\zeta_{D^2}(s)/\log(\zeta_{D^2}(0))$, then the value of s for which this function is constant as we increasing the matrix size, should give an estimate for the dimension.

In the left-hand plot of fig. 9 we have shown the spectral zeta function for the fuzzy sphere divided by $\log(N)$, where N is the number of eigenvalues of the fuzzy Dirac operator. The point of intersection is where we expect to read off the dimension. To make this more visible we plot the difference between consecutive N spectral zetas in the right hand plot. The zeroes of this can then be used as dimension estimator.

Figure 10 shows the same for the fuzzy torus. Here we see that it behaves similarly to the fuzzy sphere, despite the fuzzy torus having a spectrum that does not obey Weyl's law for the correct dimension for it's largest values.

There are a number of issues with this approach. The first is that it relies on you having a family of fuzzy spaces that are all approximating the same space. This is the case for certain highly symmetric spaces such as the 2-sphere and the 2-torus, but for the random geometries it is not clear that we are dealing with similar geometries as we increase the matrix size. The second issue is that it is a dimensional measure that requires two fuzzy spaces to compute. This is an unideal restriction as we may want to ask what dimension a certain fuzzy space is, without having to compute a sequence of fuzzy spaces just to do so. As the dimension is supposed to be an intrinsic property of each fuzzy space, requiring multiple to get an estimate is less than satisfactory. Thirdly the convergence is logarithmic in the matrix size, which is unsatisfactory as a practical dimensional measure as the random geometries are currently limited to matrix sizes of order 10. Despite its shortcomings as a dimensional measure, this procedure of heat kernel asymptotics will show itself to be very useful in determining a volume measure.

3.3 SPECTRAL DIMENSION AND SPECTRAL VARIANCE

Another dimensional measure based upon the heat kernel is the spectral dimension. This was first used in [119] as a measurement of causal dynamical triangulations (CDT), where they showed that the spectral dimension is dependent on the energy scale investigated. With the spectral dimension of a CDT being 4, the topological dimension of spacetime, at low energy scales and reducing to a value 2 at high energy scales. The notion of the dimension being a scale dependent quantity has become a popular aspect of many theories of quantum gravity - see [120] for a review. The spectral dimension has since been applied to many theories of quantum gravity, as it is easily adapted to discrete geometries as well as continuum spaces, making it an excellent comparison tool between theories.

For the squared Dirac operator, D^2, on a manifold, the spectral dimension is defined as:

$$d_s(t) := -2t\frac{d\log(K_{D^2}(t))}{dt} \tag{157}$$

It can be seen from eq. (141) that the dimension of the manifold in question can be obtained from the small t behaviour of the spectral dimensions. Specifically it can be shown that $d_s(0) = d$. As we know from the previous section, the coefficient a_0 is proportional to the volume of the space and it contributes the value d to the spectral dimension. The higher coefficients are the root of the deviation from this value as t grows, and the coefficients $a_2, a_4, \ldots$ depend on the curvature and connection. The spectral dimension has the interpretation (in quantum gravity studies) that by varying the parameter t we are probing the geometry at different energy/length scales.

If the Dirac operator has a discrete spectrum then we can express the spectral dimension in terms of the eigenvalues in the following way:

$$d_s(t) = 2t\frac{\sum_{\lambda_i}\lambda_i^2 e^{-t\lambda_i^2}}{\sum_{\lambda_i} e^{-t\lambda_i^2}}. \tag{158}$$

For large values of t the spectral variance is dominated by the smallest eigenvalues and as we decrease t the larger eigenvalues play more of a role. Thus we can view t as determining the energy scale at which the spectral dimension is measuring. With $t \to 0$ determining the infinite-energy/continuum limit. This interpretation of an energy scale dependent dimensional measure will be of interest when considering the $a = 3$ torus.

A problem with dealing with the Dirac operator is that often we do not have an eigenvector with eigenvalue zero. The constant function is always an eigenfunction of the scalar Laplacian and so the Laplacian always has a zero eigenvalue in it's spectrum. However, when the scalar curvature is non-zero the Dirac operator doesn't have a zero eigenvalue [69]. The problems arise in the following way.

If the operator has a zero eigenvalue, $\lambda_0 = 0$, then we have that:

$$d_s(t) = 2t\frac{\lambda_1^2 e^{-\lambda_1^2 t} + \lambda_2^2 e^{-\lambda_2^2 t} + \dots}{1 + e^{-\lambda_1^2 t} + e^{-\lambda_2^2 t} + \dots} \tag{159}$$

$$= \frac{2t\lambda_1^2}{e^{\lambda_1^2 t} + 1 + e^{-(\lambda_2^2 - \lambda_1^2)t} + \dots} + \frac{2t\lambda_2^2 e^{-(\lambda_2^2 - \lambda_1^2)t} + \dots}{e^{\lambda_1^2 t} + 1 + e^{-(\lambda_2^2 - \lambda_1^2)t} + \dots} \tag{160}$$

$$\sim 2t\lambda_1^2 e^{-\lambda_1^2 t} \to 0 \text{ as } t \to \infty. \tag{161}$$

However, if the magnitude of the smallest eigenvalue is non-zero, $\lambda_0 \neq 0$ we have that:

$$d_s(t) = 2t\frac{\lambda_0^2 e^{-\lambda_0^2 t} + \lambda_1^2 e^{-\lambda_1^2 t} + \dots}{e^{-\lambda_0^2 t} + e^{-\lambda_1^2 t} + \dots} \tag{162}$$

$$= \frac{2t\lambda_0^2}{1 + e^{-(\lambda_1^2 - \lambda_0^2)t} + \dots} + \frac{2t\lambda_1^2 e^{-(\lambda_1^2 - \lambda_0^2)t} + \dots}{1 + e^{-(\lambda_1^2 - \lambda_0^2)t} + \dots} \tag{163}$$

$$\sim 2t\lambda_0^2 + o(te^{-t}) \to \infty \text{ as a}t \to \infty. \tag{164}$$

So to summarise if the Dirac operator doesn't have a zero eigenvalue, then the spectral dimension grows linearly with t, in the $t \to \infty$ limit. This does not pose a problem for the manifold scenario, as we are looking at $d_s(0)$ for the dimension. However, if we plan on using this to measure the fuzzy spaces, then we may want to look at value of t greater than zero in order to avoid the behaviour due to the discreteness.

To remove this undesirable feature of the spectral dimension, the following object was proposed in [64].

Definition 33. The spectral variance is defined to be

$$v_s(t) = d_s(t) - t\frac{\mathrm{d}d_s(t)}{\mathrm{d}t} = 2t^2\frac{d^2\log(K_{D^2}(t))}{dt^2} \tag{165}$$

The spectral variance can be expressed in terms of the eigenvalues of the operator as:

$$v_s(t) = 2t^2 \left(\frac{\sum_\lambda \lambda^4 e^{-\lambda^2 t}}{\sum_\lambda e^{-\lambda^2 t}} - \left(\frac{\sum_\lambda \lambda^2 e^{-\lambda^2 t}}{\sum_\lambda e^{-\lambda^2 t}} \right)^2 \right). \tag{166}$$

Note that we still have the property that $v_s(t) \to d$ as $t \to 0$.

The origin of the name can be seen from eq. (166) as it is the variance of λ^2 in a probability distribution $p(\lambda) = \frac{1}{K} e^{-t\lambda^2}$. If we consider a system with partition function equal to heat kernel trace K, and the parameter t the inverse temperature $t = \beta = \frac{1}{T}$. Such a system can be viewed as a thermodynamical system with Hamiltonian/Energy D^2. In such a case the spectral dimension is given by the internal energy, $d_s(t) = 2t\langle D^2 \rangle$ and the spectral variance is given by the heat capacity, $v_s(t) = 2C_v$. It is worth remembering that the heat capacity of an ideal gas in flat d-dimensional Euclidean space is equal to the $d/2$. So one interpretation of this method is that we are determining the dimension of the geometry by examining the diffusion of an ideal gas (as goverened by D^2) in that geometry and measuring it's heat capacity.

An important feature of both the spectral dimension and spectral variance is that they are insensitive to overall multiplicity factors. If we double the spectrum for instance, so every eigenvalues appears twice, the spectral dimension and variance remain unchanged. This will be important when considering non-square fuzzy tori below, as the fuzzy spectrum obtained is duplicated $ad - bc$ times, see Section 2.3.2.

Also, given a Dirac operator D, we have that μD is also a Dirac operator and the eigenvalues scale in the same way. For a manifold this amounts to the distances between points scaling by a factor of μ^{-1}. Note that both the spectral dimension and spectral variance posses the scaling property:

$$d_s(\mu D, t) = d_s(D, \mu^2 t), \quad v_s(\mu D, t) = v_s(D, \mu^2 t). \tag{167}$$

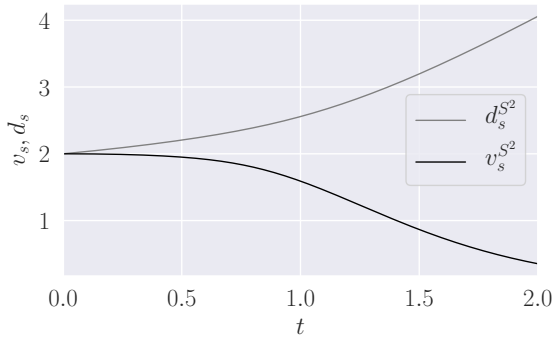


Figure 11.: Spectral Dimension and the spectral variance for the continuum sphere.

As the dimension of a manifold is independent of the distance between any two points on it[3], this scaling is deemed a trivial difference.

3.3.1 *Spectral Measures of the Fuzzy Sphere and Torus*

The fuzzy sphere and fuzzy tori are first examined to justify the use of the spectral variance as a dimensional measure. We will then take a deeper look at the random geometries.

Spheres

Figure 11 shows the spectral dimension and the spectral variance for the continuum sphere. As the 2-sphere does not have a zero eigenvalue for the Dirac operator we see the characteristic linear growth of the spectral dimension for large values of t. The spectral variances drops to zero for large t as expected. Both the spectral variance and dimension attain the value 2 at $t \to 0$, which is the topological dimension of the sphere.

Figure 12 shows the spectral dimension and spectral variance for fuzzy spheres of sizes $N = 5$ and $N = 15$. We see that for both cases the spectral dimension again grows linearly for large t, and the spectral variances goes to zero. We see that for

3 Note that the volume of the space is not independent of this behaviour.

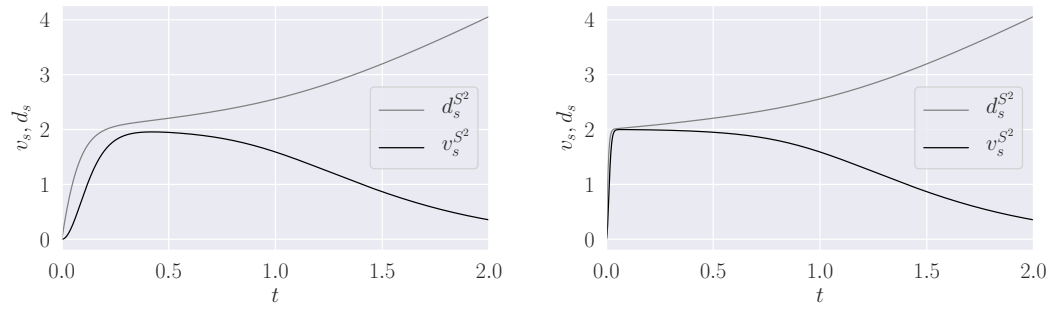


Figure 12.: Spectral dimension and spectral variance for fuzzy spheres of size $N = 5$ and $N = 15$.

the $N = 5$ fuzzy sphere, spectral dimension barely starts to plateau at the value ~ 2 for values of t between $0.1 - 0.5$, before dropping to zero as $t \to 0$. This drop to zero is attributed to the finite structure of the fuzzy sphere being seen by the measures. The spectral variance rises from zero to a value of ~ 2 for a similar range of t before dropping to zero. It is much clearer in the $N = 15$ case that both values are heading to the value 2 as we decrease the value t, before both dropping to zero.

Tori

The spectral dimension and variance for a unit square torus with spin structure $(1, 1)$ is shown in fig. 13. This spin structure is chosen to aid in the comparison between the continuum torus and the fuzzy torus as the unit square fuzzy torus has

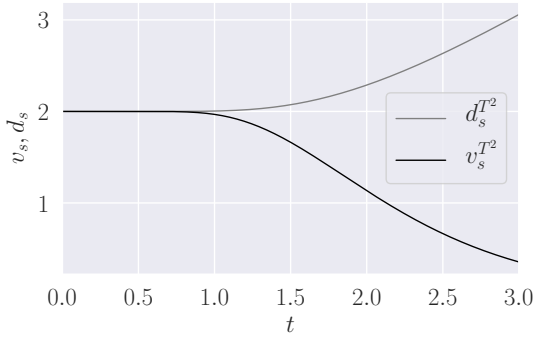
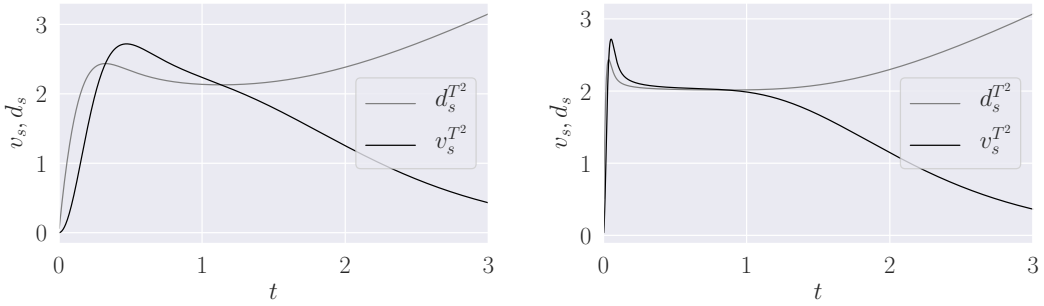


Figure 13.: Spectral dimension and spectral variance for the continuum square unit torus with spin structure $\Sigma = (1, 1)$.

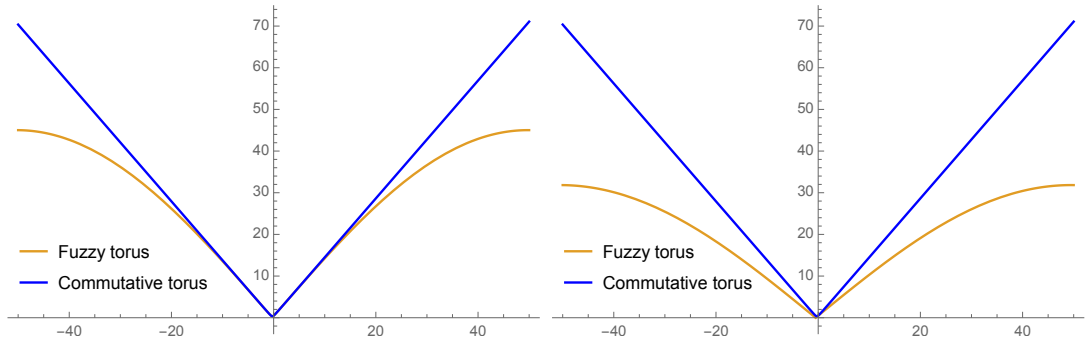


(a) $N = 10$ (b) $N = 30$

Figure 14.: Spectral dimension and spectral variance for square unit fuzzy tori of sizes $N = 10$ and $N = 30$.

a $\Sigma = (1, 1)$ spin structure. We see that for the continuum torus we see very similar behaviour to that of the continuum 2-sphere. As the Dirac operator does not have a zero eigenvalue for this spin structure the spectral dimension grows linearly for large t again and the both the spectral variance and spectral dimension attain the value 2 as $t \to 0$.

Figure 14 considers a unit square fuzzy torus. Larger matrix sizes have been used than for the fuzzy sphere as the fuzzy torus spectrum is a worse approximation to the continuum spectrum. Again the Dirac operator for this geometry does not have a zero eigenvalue, so we see that the spectral dimension grows linearly with large t. We also see that both measures fall to zero at $t = 0$ due to the finite nature, however there is a clear spike for small but non-zero t. A similar phenomenon was observed in [121] due to the discreteness in various approaches. In each case studied the discretisation causes the eigenvalues to differ from the continuum values at the largest values. This is also found for the fuzzy torus (see fig. 3). In fig. 15 we can see that there is a small region around the origin where the the fuzzy torus eigenvalues and continuum torus agree very well. But outside of this region the fuzzy torus no longer follows the linear path of the continuum torus and is more sinusoidal. Thus this spike does not seem to be a feature of the non-commutativity directly, but a feature of the discreteness. We can interpret this bump as the spectral variance



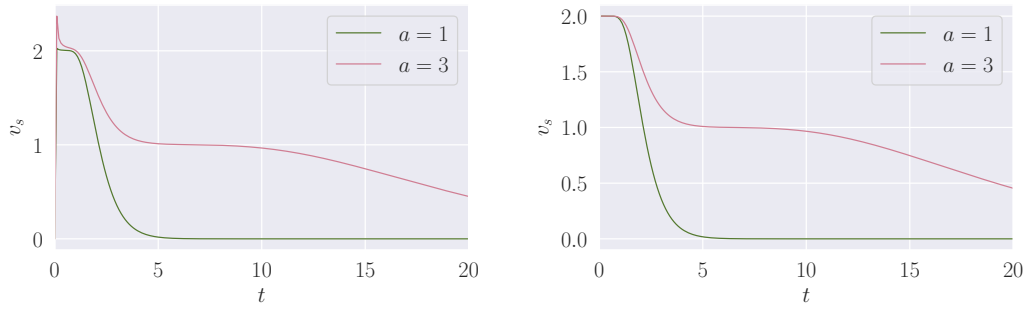
(a) $k = l$ slice (b) $l = 0$ slice

Figure 15.: Slices of fig. 3 to show how the eigenvalues deviate for large k, l. Both the continuum and fuzzy tori have spin structure $(1, 1)$ and the fuzzy torus is of size $N = 90$.

measuring the high dimensional behaviour found in fig. 5 as we probe sufficiently high energies.

For the $N = 10$, there is an inflection/saddle point of the spectral dimension that occurs at a value ~ 2 whilst the spectral variance rises from zero to a value quite above the expected value of 2 and nearly attains 3 with no real plateauing. For $N = 30$ however, the behaviour of the spectral dimension is similar to that when $N = 10$, but now the saddle point has widened and resembles a plateau and occurs at a value ~ 2, the spectral variance now also develops a plateau ~ 2 before spiking for small t. Note that the height of the spike stays roughly constant, but becomes sharper as N increases and shifts towards $t \to 0$. This is in agreement with the findings in [121, Figure 6 b)] for other discretisations of the torus (note that the figures in this paper have a logarithm scale for t, which masks the fact that the peaks sharpen as $t \to 0$).

In fig. 16 the spectral dimension and variance for two different shaped fuzzy tori are shown. Their continuum counterparts are also shown. The unit square tori are shown in green and the $a = 3$ tori are shown in pink. We have taken $N = 90$ for these fuzzy tori due to the $a = 3$ fuzzy torus requiring that N be a multiple of $ad - bc = 3$. We also need sufficiently high matrix sizes to have a well behaved spectral variance as seen in fig. 14.



(a) $N = 90$ fuzzy Tori

(b) Continuum Tori

Figure 16.: Figure (a) shown the spectral variance for two different fuzzy tori. The green line shows the unit square fuzzy torus ($a = d = 1$, $b = c = 0$) and the pink line shows the spectral variance for the $a = 3$ fuzzy torus.

The $a = 3$ torus can be viewed as identifying the opposites sides of a torus with lengths $d^2 \cdot 2\pi = 2\pi$ and $a^2 \cdot 2\pi = 18\pi$ respectively. The fact that one of the axes of the torus is nine times longer than the other has an interesting effect on the spectral variance. A clear plateau at a value of ~ 1 for relatively large t is developed. We interpret this as the rectangular torus appearing one dimensional at low energies and when using higher energies (probing a smaller structure of the space) we see the higher dimensionality. This phenomena has been noted in [121] for the spectral dimension of the Laplacian for differently shaped tori.

It should be noticed that the spike at low values of t is more pronounced and wider for the $a = 3$ torus than for the unit square torus. This is due to the fact that we comparing them both at $N = 90$ for both cases, where the $a = 3$ fuzzy torus has a 3-fold spectral degeneracy effectively reducing it to a $N = 30$ approximation. The 3-fold degeneracy does not have an effect on the spectral variance as it is insensitive to overall multiplicity factors as was discussed earlier. Figure 17 shows the comparison between then spectral dimension for the $N = 30$ fuzzy square unit torus and the $N = 90$ fuzzy $a = 3$ torus. It can be seen that the spike at low values of t for both cases now are identical in width and height. This shows that non-square fuzzy tori are worse approximations to the continuum than the unit square tori for a given matrix size.

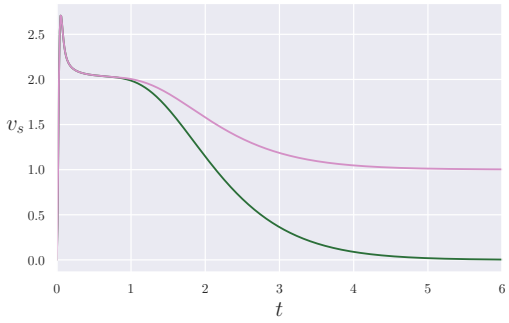


Figure 17.: Spectral dimension for the square unit torus of matrix size $N = 30$ and the fuzzy $a = 3$ torus with $N = 90$. Showing that the non square tori are worse approximations to the continuum and that the breadth and height of the spike at low values of t is tied to the accuracy of the approximation.

The proposal is thus, to use the spectral variance of a Dirac operator as a measure of the dimension of a fuzzy space. This requires some justification as it is not always clear what the dimension of a non-commutative space means. There are however a number of properties we require of the dimension. For instance, if the spectrum approximates the spectrum of a manifold (such as in the cases above), then the dimension should agree with the topological dimension of the manifold up to an appropriate level of approximation. For a manifold, the spectral dimension and variance both measure the dimension of the space as $t \to 0$, whereas for fuzzy spaces we require to look at non-zero values of t. From the examples investigated above, it is clear that we want to investigate where the spectral variance plateaus, i.e. has an interval of t for which the spectral variance is approximately constant. These plateaus can be viewed as mimicking the behaviour of flat space and show there is an energy-scale where the spaces appear to have manifold behaviour.

The spectral variance of the fuzzy sphere has just one stationary point and is therefore the maximum of the graph. The value of the spectral variance at this maximum agrees well with the continuum dimension, 2. For the torus the graph shows surprisingly good agreement with the continuum spectral variance, and the value 2 for a wide region, despite the fact that the actual spectra are substantially different. The only marked difference is that the fuzzy torus has a higher peak at very small t. This

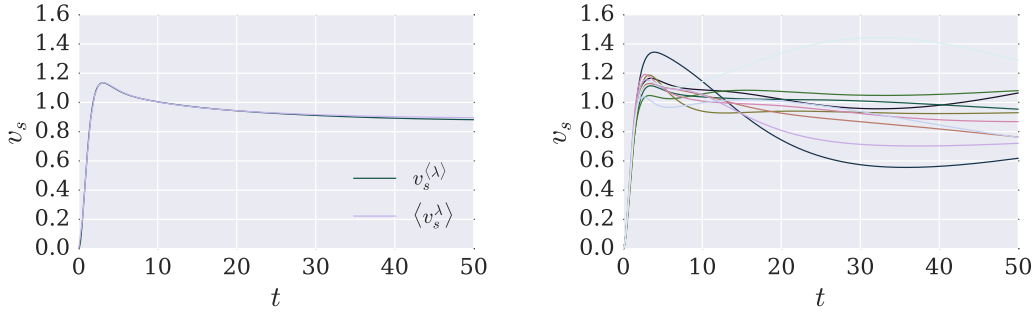
peak arises because the large eigenvalues of the fuzzy torus are much smaller than those of the continuum torus, and start to follow a more sinusoidal path, which is seen as following a Weyl's law for a much higher dimension, see fig. 5. This example makes it clear that it is not the global maximum that is important, rather it is the larger region of t for which the graph is approximately constant that matters.

3.3.2 *Spectral Variance of Random Geometries*

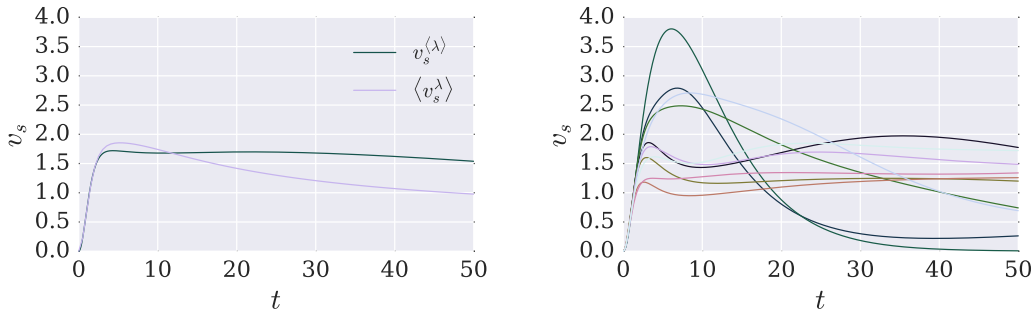
With the analysis of the above section in mind, the spectral variance of the random fuzzy spaces are now examined. The geometries of type $(2,0)$ and $(1,1)$ were examined in [61, 63], so it is interesting to add the understanding of their dimension. The most important difference found in [63] is that the phase transition for the type $(1,1)$ leads to much weaker correlations. The accompanying shift in behaviour is much more gradual than for type $(2,0)$, hence no large jumps between different g_2 values for the type $(1,1)$ geometries are expected.

As mentioned in section 2.3.3 we have two options for calculating the spectral variance. We can calculate the ensemble average of the spectral variance, or alternatively we can calculate the spectral variance of the averaged eigenvalues. We compare both methods in fig. 18 for the type $(2,0)$ geometries. Note that the range of t values in the plots is much larger than for the fuzzy sphere or fuzzy torus case. This is due to the fact that the spectrum for the random geometries is bounded above by a low number ~ 2.

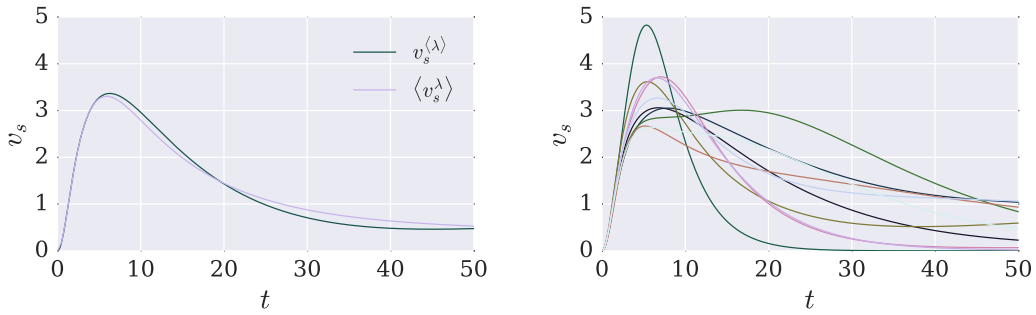
Note that for values of g_2 away from the phase transition, the spectral variance from the averaged eigenvalues and the ensemble average of the spectral variances are very similar. While at the phase transition these two definitions produce slightly different graphs. This is illustrated in the left-hand plots in fig. 18. The right-hand plots of fig. 18 show the spectral variance for a sample of geometries taken from the respective ensembles. This shows that the geometries that contribute to one ensemble can be very different, particularly close to the phase transition. From the



(a) Type $(2,0)$ $g_2 = -2.5$ $N = 10$



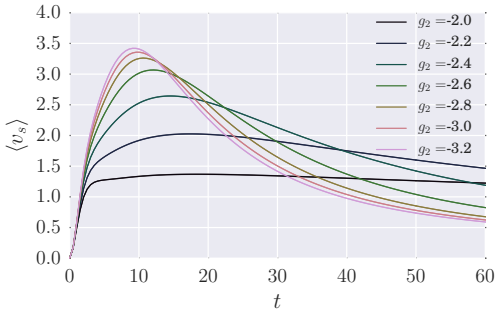
(b) Type $(2,0)$ $g_2 = -2.8$ $N = 10$



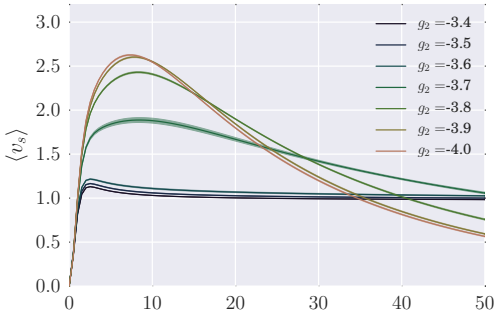
(c) Type $(2,0)$ $g_2 = -3.5$ $N = 10$

Figure 18.: The left-hand plots are comparing the spectral variance as calculated from the average eigenvalues, $v_s^{\langle\lambda\rangle}(t)$ with the average spectral variance $\langle v_s^\lambda(t)\rangle$. In figure (a) the two lines are so similar that they almost can not be distinguished in this plot. The right-hand plots are showing 10 uncorrelated examples of spectral variances from the ensemble of geometries.

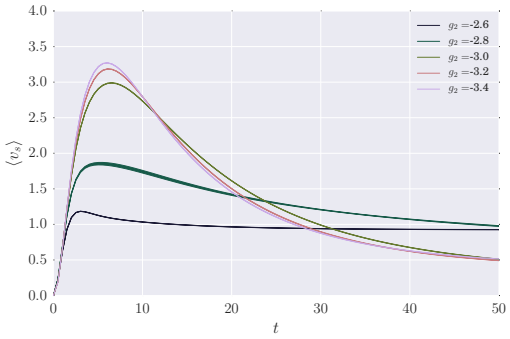
graphs one can see that the variance of the spectral variance at a given value of t is large, i.e. of order 1, at $N = 10$. A good question is whether this variance decreases for larger matrix sizes. This will need further data to determine. Looking at the



(a) $(1,1)$ $N = 10$



(b) $(2,0)$ $N = 10$



(c) $(1,3)$ $N = 8$

Figure 19.: The spectral variance of the averaged eigenvalues for the three types of random geometry studied as the g_2 value is varied, at the maximum matrix size. Showing that the spectral variance undergoes a rapid change of behaviour around the phase transition value.

plots in detail, the curves in fig. 18 (a) are qualitatively similar to each other, as are the ones in fig. 18 (c). However at the phase transition, in fig. 18 (b), one sees curves similar to both (a) and (c), suggesting that the system spends some time on each side of the transition.

In fig. 19 the spectral variance of the averaged eigenvalues for the three types of random geometry are shown for various values of the coupling constant g_2 in the action. The maximum matrix size available is used in an attempt to get behaviour that is most like a continuum geometry.

A common trend for the plots is that before the phase transition the spectral variance it is mostly flat and around the value ~ 1 and then rapidly grows at the phase transition and becomes more peaked at the value of g_2 grows. The type $(1,1)$ ge-

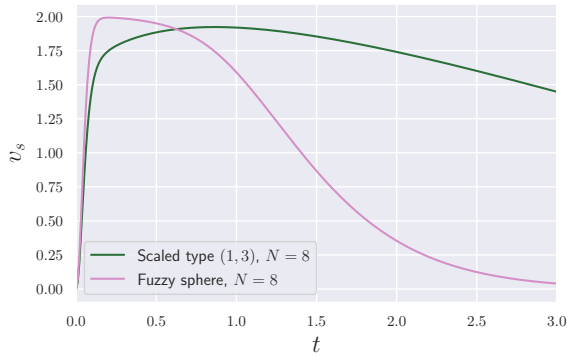


Figure 20.: The spectral variance for the average spectrum of the type $(1,3)$ random geometry at the phase transition value of $g_2 = -3.70$ at $N = 8$ compared with the $N = 8$ fuzzy sphere. The averaged spectrum rescaled so the maximum eigenvalue agrees with the maximum of the fuzzy sphere spectrum of the same matrix size.

ometry behaves slightly differently to the other types, which indicates that its phase transition is of a different type to the $(2,0)$ and $(1,3)$ as was hinted at in [63]. For the type $(2,0)$ and $(1,3)$ geometries, we see that the spectral variance has a maximum close to the value 2 for g_2 values near the phase transition, which agrees with the behaviour found in [61, 63].

The fuzzy sphere introduced in [46] is a type $(1,3)$ fuzzy space, the same as the random type $(1,3)$ geometries. In order to compare the random geometries to the fuzzy sphere, we need to rescale the eigenvalues in an appropriate manner. In fig. 20 the spectral variance is compared for the type $(1,3)$ random geometry at the phase transition and fuzzy sphere, both with matrix size $N = 8$. The averaged spectrum of the random geometry is rescaled so the maximum eigenvalue matches the maximum eigenvalues of the fuzzy sphere. While the $(1,3)$ geometry and the fuzzy sphere do not agree completely, they behave in a similar way. For the random geometries the spectral variance rises slowly for small t. This is due to the fact that the density of eigenvalues for the fuzzy sphere linearly rises for the near entirety of its spectrum. However, the density of eigenvalues for the random fuzzy spaces experiences as drop as we look at higher eigenvalues - as was shown in [61]. For the random geometries the measures also experience a slower decay than they do the

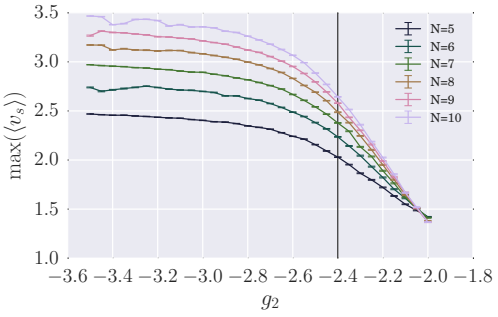
fuzzy sphere as t becomes large. This is because the lowest eigenvalue of the random fuzzy geometries is typically much smaller than that of the fuzzy sphere, even after rescaling. The reason for this is that the fuzzy sphere has degenerate eigenvalues due to the spherical symmetry, while for a generic random fuzzy space the eigenvalues have the minimal allowed multiplicity (which is 2 for $(1,3)$ geometries, as explained in [61]), to maximise the entropy.

For type $(2,0)$, the maximum of the spectral variance rises with lowering g_2, just as for type $(1,3)$. As shown in fig. 19 (b), the spectral variance near the phase transition ($g_2 = -2.8$) has a maximum value close to 2 and is qualitatively similar to the fuzzy sphere of a similar matrix size. This suggests a 2-dimensional geometry at the phase transition but more work would be needed to substantiate this. For type $(1,1)$, the spectral variance close to the phase transition ($g_2 = -2.4$) reaches values above 2 and the overall shape of the curve is very different from that of the fuzzy sphere. In particular, the curves just below and just above the phase transition are not as different from the curve at the phase transition as they are for type $(2,0)$. This is shown in fig. 19 (a).

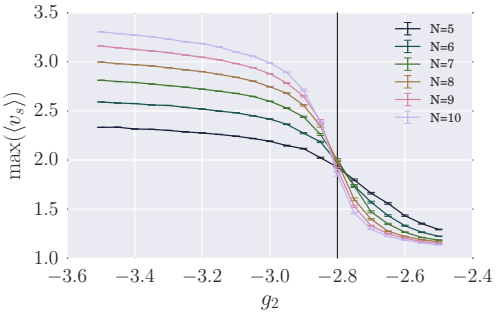
These plots show that the tentative conclusion of [61] that the geometries behave similarly does not survive more detailed examination. It confirms the differences found in [63] but remains purely qualitative. To make quantitative judgements one needs more tools, like the zeta-function distance to be introduced in section 3.5.

3.3.3 *The maximum of the spectral variance*

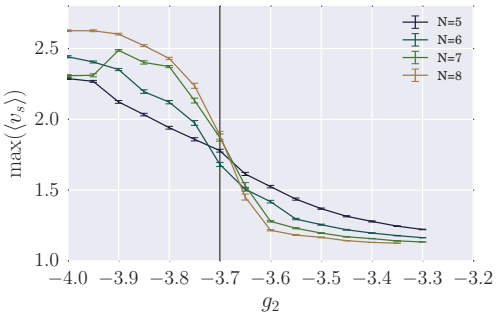
The average spectral variance curve is zero at $t = 0$ and $t \to \infty$ and so has a maximum value. There has been only one local maximum in all of the cases of random geometries studied here. This maximum value $\max(\langle v_s \rangle)$ is therefore a very crude estimate of the dimension. The limitation of this approach is that the maximum varies widely within each ensemble and so the interpretation is not so clear. Nevertheless, it still proves instructive to plot $\max(\langle v_s \rangle)$ for the random geometries.



(a) Type $(1,1)$



(b) Type $(2,0)$



(c) Type $(1,3)$

Figure 21.: The maximum of the average spectral variance plotted against g_2. The vertical lines indicate the phase transition values as determined in [63].

Plotting the maximum value against g_2 leads to the curves in fig. 21. The separate lines in that figure are for different values of the matrix size N. For fixed N, the maximum rises as g_2 becomes more negative. It starts out around 1 and then rises to large values after the phase transition. A particularly interesting feature is that for types $(2,0)$ and $(1,3)$ the maximum of the spectral variance around the phase transition seems to be close to 2 independent of N. An optimistic interpretation of this would be that the behaviour at the phase transition shows a certain scale freedom and might remain the same in the large N, continuum, limit. For type $(1,1)$ on the other hand the point of intersection seems to lie around $g_2 = -2.05$ and has a value of about 1.5. This g_2-value is considerably below the phase transition point determined in [63], however of the three geometries examined type $(1,1)$ had the least clear signal at the phase transition, which makes the determination of the

phase transition in this case less certain. A priori, no reason exists to expect all of the spectral variances to cross in close proximity. Therefore this intersection is another marker of interesting behaviour at the phase transition.

3.4 VOLUME OF A FUZZY SPACE

The volume of a Riemannian manifold, (M, g) is given by the following expression:

$$Vol(M, g) = \int_M \sqrt{-g} \mathrm{d}^d x \tag{168}$$

and as such it inherently depends on the metric chosen. This way to define the volume also requires you to know the dimension of the manifold. Note that this notion of "a volume" generalises the ordinary definition of volume for a three dimension space to all dimensions, i.e. in 2-dimensions it is the area etc.

As spectral triples are to be thought of as the algebraic data for a compact Riemannian manifold, there should be a notion of the volume of a spectral triple. As the focus of this thesis is to look at spectral triples using just the spectrum of their Dirac operators. This section is dedicated to investigating the different methods to calculate the volume of a given geometry using the spectrum of a Dirac operator.

3.4.1 *The Dixmier Volume*

As the notion of a metric tensor does not survive the transition to noncommutative geometry, other methods than performing the integral above need to be considered. It was highlighted in the previous section that the heat kernel of pseudo-differential operators defined over a compact Riemannian manifold contains the volume information in its asymptotic expansion. This volume can be extracted by looking at the residue of the rightmost pole of the spectral zeta function in eq. (147). For a fuzzy space, there are a finite number of eigenvalues, and hence the sum in eq. (144) is

finite, which automatically regularises the poles. However, as was discussed, there is a method to locate the potential location of the right-most pole of the limiting geometry when we take $N \to \infty$ and recover a continuum geometry.

This was done by examining the finite steps in the Dixmier trace. The Dixmier trace is a singular trace, i.e. it vanishes on operators of finite-rank (such as the Dirac operators in a finite-spectral triple) and is described by a generalised limit procedure. However for positive, compact operators, T whose ordered eigenvalues, $\{\lambda_i\}_{i=1}^{\infty}$ grow as as $\lambda_n \sim o(n^{-1})$ then we have the following expression:

$$Tr_{Dix}(T) = \lim_{j\to\infty} \frac{1}{\log(j+1)} \sum_{i=1}^{j} \lambda_i \tag{169}$$

To make the link with extracting the heat kernel expansion coefficients the following theorem can be used:

Theorem 6 ([65] Proposition 4 p.306)**.** *Let $\mu_n(T)$ denote the n-th eigenvalues of an operator T increasingly ordered. For positive, compact operators, T, whose order eigenvalues, $\{\lambda_i\}_{i=1}^{\infty}$ grow as*[4] *$o(n^{-1})$ the following are equivalent:*

- $(s-1)\zeta_T(s) \to L$ *as* $s \to 1^+$
- $\frac{1}{\log(j+1)} \sum_{i=0}^{j} \mu_n \to L$ *as* $j \to \infty$

where $\zeta_T(s) := Tr(T^s) = \sum_{n=0}^{\infty} \mu_n(T)^s$

Examples of the operator T can be built from the Dirac operator or the Laplacian on a manifold. So if we were to consider the Dirac operator on a compact Riemannian spin manifolds, we would need to consider $|\not{D}|$ to make it positive. By using Weyl's law for Dirac operators we have that $\mu_n(|\not{D}|) \sim c(d)n^{1/d}$, where $c(d)$ is constant with respects to n. Thus we have to consider the operators $|\not{D}|^{-ds}$ to get an

4 An operator with this property is called an infinitesimal of order 1.

example of T. So we have that $(s-1)\zeta_{|\not{D}|}(s) = (s-1)Tr(|\not{D}|^{-ds})$ this gives us the following formula:

$$Tr_{Dix}(|\not{D}|^{-ds}) = \lim_{s\to 1}(s-1)Tr(|\not{D}|^{-ds}) = \frac{1}{d}\lim_{t\to d}(t-d)Tr(|\not{D}|^{-t}) \tag{170}$$

If we have instead the Laplacian (or the square of the Dirac operator $\not{D}^2$) we have that $\mu_n(\Delta) \sim c(d)n^{2/d}$, so to get an infinitesimal of order 1 we need to examine: $\Delta^{-d/2}$ the formula changes to the following:

$$Tr_{Dix}(\Delta^{-\frac{d}{2}}) = \lim_{s\to 1}(s-1)Tr(\Delta^{-ds/2}) = \frac{2}{d}\lim_{t\to d/2}(t-d/2)Tr(\Delta^{-t}) \tag{171}$$

Euler's gamma function $\Gamma(s)$ only has poles at the negative integers $s \in \{-1,-2,\dots\}$, so for $k \leq d$ we can rewrite eq. (146) to be the following:

$$a_k = \Gamma(\frac{d-k}{2})Res_{s=\frac{d-k}{2}}\zeta_{D^2}(s). \tag{172}$$

It is assumed that

$$b_j(d/2) = \frac{1}{\log(j+1)}\sum_{i=1}^{j}(\lambda_i^2)^{-d/2} \tag{173}$$

is a good approximation for $2/d$ times the residue of the pole. Note that this requires you to know the dimension of the space. It is possible to use this expression for the Dixmier trace to give a new definition of a volume from a finite number of eigenvalues, providing there is already an estimate d of the dimension,

$$Vol^{\mathrm{Dix}}(D) = \frac{d}{2}\frac{(4\pi)^{d/2}}{k}\Gamma\left(\frac{d}{2}\right)\frac{1}{\log j}\sum_{\lambda}(\lambda_n^2)^{-d/2}\,. \tag{174}$$

For a manifold, the sum is over the j smallest eigenvalues and this gives the volume approximation considered above. For Dirac operators with a finite number of eigenvalues (e.g., a fuzzy space), the sum is over all j eigenvalues and this gives a definition of a volume measure for noncommutative spaces.

This expression obeys the scaling property that for $\mu \in (0, \infty)$

$$Vol^{\text{Dix}}(\mu D) = \mu^{-d} Vol^{\text{Dix}}(D) \tag{175}$$

that also holds for the volume of a Riemannian manifold. However it is not additive,

$$Vol^{\text{Dix}}(D_1 \oplus D_2) \neq Vol^{\text{Dix}}(D_1) + Vol^{\text{Dix}}(D_2) \tag{176}$$

in general. It is not additive even if $D_1 = D_2$, since $\log(2j) \neq \log j$. In this case, $D = D_1 \oplus D_1$ just doubles the multiplicities of the eigenvalues of D_1. Thus the definition is not consistent because taking two particles on a fuzzy space (which means doubling the spinors and replacing k with $2k$) would lead to a change in the volume.

3.4.2 *The Stern Volume*

Due to the work of Stern [66], a new method for extracting the heat kernel coefficients is available. It is similar to that of the Dixmier trace, in that it expresses the residues of the spectral zeta function as a limit of partial sums of particular functions of the elliptic operator. It then utilises the same asymptotic expansion (141) as was discussed above to relate these partial sums to geometric quantities. Partial sum expressions for not only the rightmost pole of the spectral zeta functions, but also the lower order poles were developed, culminating in the following theorem.

Theorem 5 ([66] Theorem 1)**.** *For any finite set $\{s_i\}_{i=0}^{k}$ of decreasingly ordered reals and for all elliptic operators, D with:*

- *Tre^{-tD} possesses an asymptotic expansion as $t \to 0^+$ of the form $\sum_{i=0}^{\infty} c_i t^{-s_i}$, where the set $\Pi = \{s_i\}$ are decreasingly ordered reals.*
- $\Pi \cap [s_k, \infty) \subset \{s_i\}_{i=0}^{k}$

then there exists a function F such that

$$Res_{s=s_k}\Gamma(s)\zeta_D(s) = \lim_{\Lambda\to\infty} \varepsilon(\Lambda)^{s_k} \sum_{\substack{\lambda\in Spec(D)\\ \lambda<\Lambda}} F(\lambda\varepsilon(\Lambda)) \tag{177}$$

for $\epsilon(\Lambda) := m\log(\Lambda)/\Lambda$ *for any* $m > s_0 - s_k$.

However, it is shown in [66] that for our purposes, that if we find a function f such that:

- f is piecewise continuous and supported in $(1,\infty)$
- $f(t)$ decays rapidly as $t\to\infty$, specifically it is of order $O(t^{-m})$ for all $m\in\mathbb{R}$ as $t\to\infty$
- $\int_o^\infty t^{-s_i} f(t)dt = 0$ for $i < k$
- $\int_0^\infty t^{-s_k} f(t) = 1$

then it's Laplace transform

$$F(s) = \int_0^\infty e^{-ts} f(t)dt \tag{178}$$

is such a function required in theorem 5 for the coefficient $a_k = Res_{s=s_k}\Gamma(s)\zeta_D(s)$.

For our purposes by taking $f = e^{-t}$ for $t\geq 1$, so that $F(s) = e^{-1-s}/(1+s)$, and then normalising[5] by $\int_0^\infty t^{-s_0} f(t)dt = \Gamma(1-s_0,1)$ as to satisfy the above properties. We have the following:

Corollary 1. *If D is as in the above theorem with set* Π *bounded above by* $s_0\in\mathbb{R}$ *and if* $\varepsilon(\Lambda) = \log(\Lambda)/\Lambda$ *we have that:*

$$Res_{s=s_0}\Gamma(s)\zeta_D(s) = \lim_{\Lambda\to\infty} \frac{\varepsilon(\Lambda)^{s_0}}{e\Gamma(1-s_0,1)} \left(\sum_{\substack{\lambda\in Spec(D)\\ \lambda<\Lambda}} \frac{e^{-\lambda^2\,\varepsilon(\Lambda)}}{1+\lambda^2\,\varepsilon(\Lambda)} \right) \tag{179}$$

5 $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ is the upper incomplete gamma function.

The importance of this is it allows us to define a new volume measure based off of a finite number eigenvalues, recovering the continuum definition in the infinite limit. For the Dirac operator, D, we consider the above corollary for D^2 and eq. (147):

$$Vol^{\mathrm{St}}_{\Lambda}(D) = \frac{(4\pi\epsilon(\Lambda))^{d/2}}{ek\Gamma(1-\frac{d}{2},1)} \sum_{\substack{\lambda\in Spec(D)\\ \lambda<\Lambda}} \frac{e^{-\lambda^2\,\varepsilon(\Lambda)}}{1+\lambda^2\,\varepsilon(\Lambda)} \tag{180}$$

where $\epsilon(\Lambda) = (\log\Lambda)/\Lambda$. Stern's result is then that for the Dirac operator of a manifold, $Vol^{\mathrm{St}}_{\Lambda}(D) \to Vol(M)$ as $\Lambda \to \infty$. This version of the volume converges faster than the usual Dixmier trace, as it's remainder term is of order $O(\Lambda^{s_1-s_0}(\log(\Lambda))^{s_0-s_1})$ whilst the remainder for the Dixmier trace is of order $1/\log(N)$.

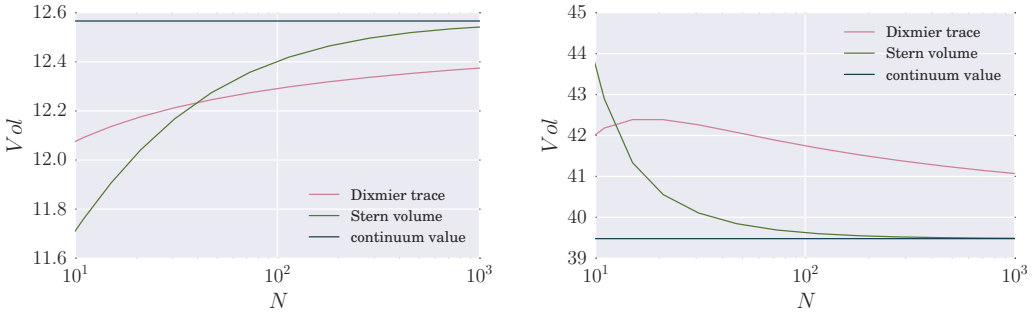
Note that Λ is a free parameter, and so when considering fuzzy spaces, it need not coincide with the maximum eigenvalue available. It is not necessary for the definition to insist that $\Lambda \geq |\lambda|_{\max}$ for all eigenvalues either. For compatibility with the original formula Λ may be taken to be either the maximum $|\lambda|$ or possibly an estimate for it. This could be determined by a dimensionful coupling constant in the action for a random fuzzy space, for example ${g_4}^{-1/4}$ in eq. (122). This may also prove useful when considering examples such as the fuzzy tori, whose spectrum is wildly non-manifold like for large eigenvalues. By tuning this parameter we could get a good convergence. However there is no systematic procedure developed yet.

The formula eq. (180) does not have the scaling property with Λ fixed, i.e.

$$Vol^{\mathrm{St}}_{\Lambda}(\mu D) \neq \mu^{-d} Vol^{\mathrm{St}}_{\Lambda}(D), \tag{181}$$

as one might expect since Λ determines a maximum eigenvalue to consider. The scaling property could be restored by simultaneously changing the value of Λ. The formula is, however, additive providing that the dimension d is the same for both geometries. In such cases one has

$$Vol^{\mathrm{St}}_{\Lambda}(D_1 \oplus D_2) = Vol^{\mathrm{St}}_{\Lambda}(D_1) + Vol^{\mathrm{St}}_{\Lambda}(D_2). \tag{182}$$



(a) Fuzzy sphere

(b) Fuzzy unit square torus

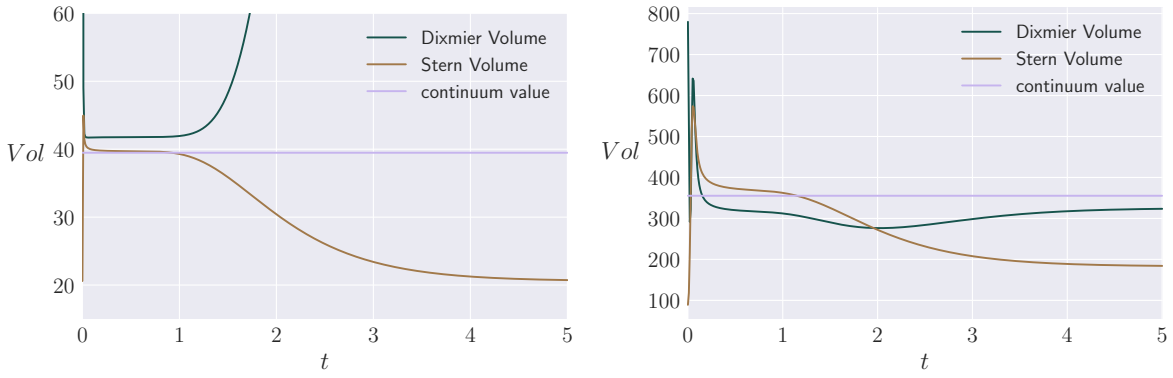
Figure 22.: Volumes of the fuzzy sphere and fuzzy unit square torus. The volumes of the continuum geometries are included for comparison.

3.4.3 *Application to fuzzy spaces*

The Fuzzy Sphere and Fuzzy Tori

Equations (174) and (180) can be used to calculate the volume of a fuzzy space, assuming the dimension is known. To test these expressions, they can be applied to the spectra of the fuzzy sphere and the fuzzy torus using $d = 2$ and Λ as the maximum eigenvalue. This is shown in fig. 22, together with the volume of the continuum sphere and the continuum torus. In the figure legend, eq. (174) is referred to as the Dixmier volume and eq. (180) as the Stern volume. As expected, the Stern volume converges much faster with N.

If the dimension parameter is not known, it can be estimated using the spectral variance at a given value of the parameter t, which determines an energy scale. The volume measures of the two fuzzy tori are shown in fig. 23. In this figure the matrix size is kept fixed at $N = 90$ and the dimension is taken to be the value of the spectral variance at the value t, i.e. $d = v_s(t)$. The plot is then shown against the parameter t. As one would expect, this agrees with the values of the volume shown in fig. 22 (b) when the spectral variance (fig. 16) is close to the value 2, which is approximately the region $0.2 < t < 1$. It also shows that the volume changes away from this quite rapidly as the dimension estimator changes.



(a) Fuzzy unit square torus

(b) Fuzzy $a = 3$ torus

Figure 23.: The volume measures of the two different tori considered are plotted against t, using the dimension estimator $d = v_s(t)$ and $N = 90$ for both.

The Random Geometries

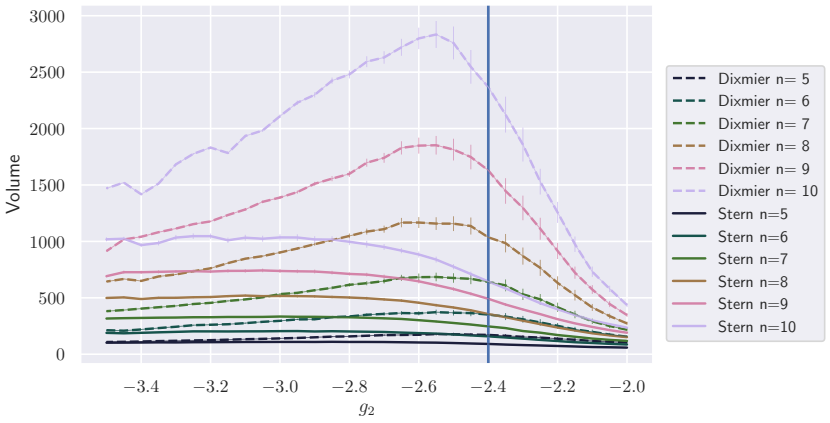
Testing the volume measures on random fuzzy geometries leads to several methodological conundrums. The first of these is the question of dimension. For the torus and the sphere their topological dimensions are known, and can be used to calculate their volume; no such information is available about the random fuzzy spaces. One could use the spectral variance defined above to define the dimension in the volume, but this is scale dependent. Even more troubling is that in the ensembles of random geometries used in this work, the spectral variance fluctuates substantially, as can be seen in fig. 18 on the right. The maxima of the spectral variances of the individual Dirac operators have a large variance (of order 1), and moreover, the value of t for these maxima also has a large variance. The question is, how could one average over the volumes of different geometries if these geometries are of different dimension? The average volume of a cube, a square and a line makes no physical sense, since their volumes are different quantities. There are possible solutions to this problem, such as measuring the volume of an average geometry, e.g. using the left hand spectral variance curves in fig. 18. This could lead to a unique number for the volume but it is not clear if that usefully represents the properties of the ensemble.

The second issue arises due to the scaling that occurs in the action eq. (122). This was mentioned in the context of dimensions and was shown not to affect the measures there. However the same cannot be said for the volumes. As the maximum eigenvalue of the random geometries is heavily constrained to be ~ 2 for every matrix size. This means that comparing the volumes for different matrix sizes is unlikely to produce meaningful results. However it still could provide an insight into the overall behaviour of the geometries as the coupling constant g_2 is varied. To combat this scaling we make use of the knowledge expressed in eq. (175), that the volume measure based upon the Dixmier trace scales inversely proportional to the scaling imposed on the Dirac operator. Thus we will get a scaling dependent volume from the Dixmier volume. We can force a Weyl's law behaviour to the eigenvalues by multiplying the spectrum by $N^{1/d}$, where N is the total number of eigenvalues available and d is the dimension as determined by the maximum of the spectral variance. This will force the largest eigenvalue to obey a Weyl's law scaling, however the lower eigenvalues need not follow the same scaling depending on their distribution. Again this has conceptual problems as the dimension of these spaces fluctuates.

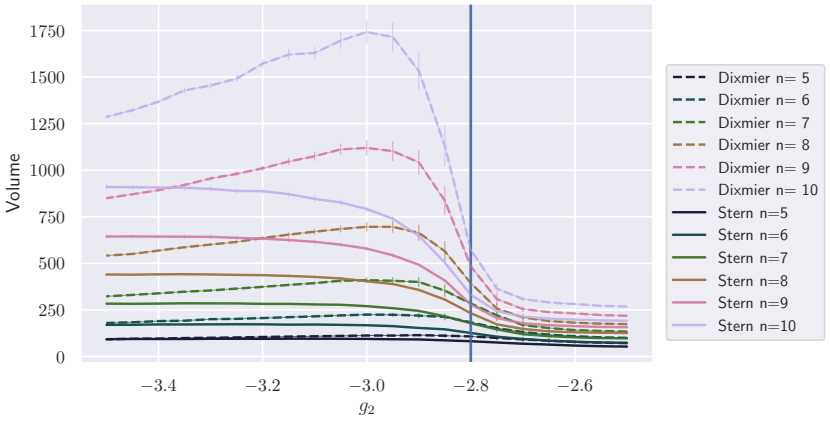
The volume of the averaged eigenvalues for the random geometries are investigated in fig. 24. The dashed lines represent the Dixmier volume and the solid lines represent the Stern volume for the different matrix sizes available. In both cases the maximum of the spectral variance is used as a measure of the dimension. For the Stern volume, Λ is taken to be the maximum eigenvalue, so we take into account the entire spectrum available.

It can be clearly seen for the type $(2,0)$ and type $(1,3)$ geometries that the phase transition marks a pivotal point in the behaviour of the simulation. Before the phase transition (to the right of the blue line in the plots) the volumes are fairly stable for the range of g_2, then at the phase transition value the volumes grow rapidly.

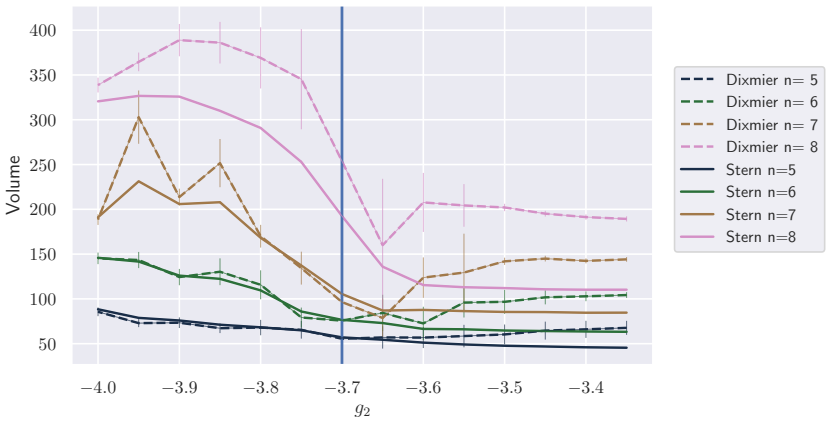
For the type $(1,1)$ the volumes behaved differently, with the growth of the volumes starting before the phase transition and continuing to grow until just after. This behaviour is tied to the behaviour of the maximum of the spectral variance as shown in fig. 21.



(a) Type $(1, 1)$

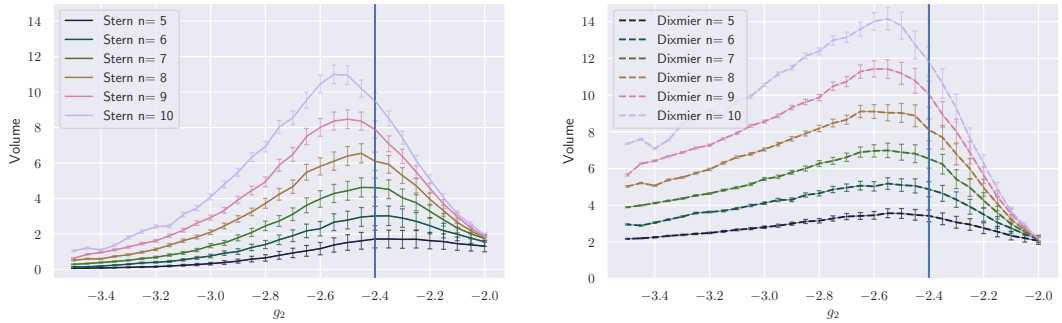


(b) Type $(2, 0)$

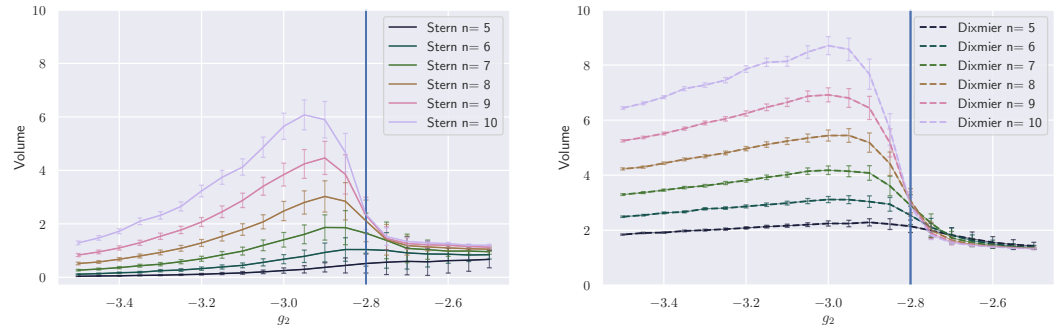


(c) Type $(1, 3)$

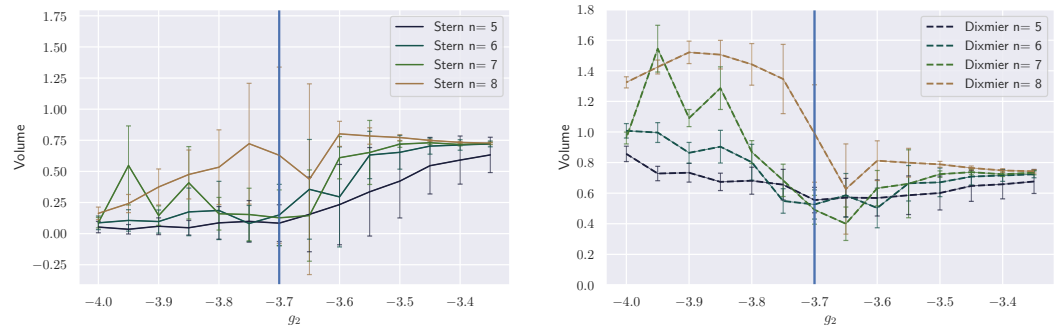
Figure 24.: The Dixmier (dashed line) and Stern (solid line) volumes for the random geometries. The averaged eigenvalues are used and the uncertainties are propagated and shown.



(a) Type $(1,1)$



(b) Type $(2,0)$



(c) Type $(1,3)$

Figure 25.: The Stern and Dixmier volumes for the random geometries with the averaged eigenvalues multiplied by $N^{1/d}$. The uncertainties are more pronounced so each volume has its own plot, the Stern volume is the left hand plots and the Dixmier volume on the right.

The volumes of the geometries appear to grow with the matrix size, this is due to the scaling that occurs in the action. To remove this feature we rescale the eigenvalues by $N^{1/d}$ as mentioned above, where N is the total number of eigenvalues. The resulting volume measures are shown side by side in fig. 25. As a result of the scal-

ing, the volume measures now take values in a much smaller range than compared to the unscaled versions in fig. 24. The growth of the volumes with the matrix size has been drastically reduced before the phase transition. The phase transition for type $(2,0)$ and $(1,3)$ marks the sudden growth of the volumes, similarly to fig. 21, where the phase transition marks the sudden growth of the maximum of the spectral variance. One thing to note is that the uncertainties on the volume measures become much more pronounced after the scaling. With the uncertainties for the type $(1,3)$ geometry being so large for the Stern volume that it is difficult to make any concrete claims. However, the data available for the type $(1,3)$ random geometries is worse. This is due to the increased dimension of its Clifford module, resulting in more matrices H_i, L_j in matrix model, and the action therefore becoming much more complicated as the Dirac operator has more terms. This was discussed in [63].

However, the type $(2,0)$ geometries has the clearest indicator of a phase transition in [63] and has low uncertainties associated to the averaged spectra. This geometry also shows the clearest behaviour, with the volume being roughly constant before the phase transition once the matrix scaling has been accounted for. At the phase transition we see the volume measures begin to spike. The behaviour of the Stern and Dixmier volumes then differs slightly after the phase transition, with the Stern volume reaching a maximum just after the phase transition and then decays to zero volume. The Dixmier volume however reaches a maximum at the same value of g_2 however the decay is much slower. This behaviour can be seen in the type $(1,1)$ geometry plots also. However, the uncertainties in the Stern volume for the type $(1,3)$ geometries are too wild to make any conclusions. With the improvement of the simulation speed, larger matrices and more geometries can be examined for each geometry, which would be needed to examine the larger Clifford type geometries.

The volumes of the random geometries is then a difficult object to calculate, as it depends heavily on the dimension you choose. Using the maximum of the spectral variance as the dimension inherently links the volume to the behaviour of the dimension. Thus the information gained by examining the volumes measures is not independent from the information gained from examining the dimension. However

it is still an interesting fact that the volumes appear to change behaviour, along with the spectral variance, at the phase transition. For future investigations the spectral variance and volume measures could be used in the action potential as they are spectral. Thus it could be possible to fine tune the simulation to give geometries with specific dimensions or volumes, and fluctuate other geometries quantities.

3.5 THE ZETA DISTANCE

We introduce one final spectral measure to compare geometries that is equally well defined in both the continuum and fuzzy geometry cases.

In [67] a distance between two geometries was defined using the ratio of the spectral zeta functions of the Laplace-Beltrami operator. This definition is adapted here to use Dirac operators instead. If we let D_1, D_2 be Dirac operators and ζ_1, ζ_2 their zeta functions, as in eq. (144). A real number γ is chosen so that it is greater than the real part of any pole of either zeta function.

The zeta distance between the geometries is defined to be

$$\sigma(D_1, D_2) = \sup_{\gamma \leq s \leq \gamma+1} \left| \log \left(\frac{\zeta_1(s)}{\zeta_2(s)} \right) \right|. \tag{183}$$

Note that the zeta functions are positive for the relevant values of s. The distance has the property that $\sigma(D_1, D_2) = 0$ if and only if the spectra are the same [67]. It also obeys the triangle inequality

$$\sigma(D_1, D_2) + \sigma(D_2, D_3) \geq \sigma(D_1, D_3). \tag{184}$$

The definition uses the closed interval $[\gamma, \gamma + 1]$ but the 1 is just for convenience and in fact any finite interval with a suitable lower limit γ will do. The definition also holds for finite spectra, in which case there is no restriction on γ. It can even be used to compare a finite spectrum to an infinite one. Several examples are examined in detail in the following sections.

3.5.1 *Convergence of fuzzy spectra to the continuum*

The distance measure can be used to define the convergence of spectra. If $\sigma(D_n, D) \to 0$ for a sequence of Dirac operators $\{D_n\}$, then $\zeta_n(s) \to \zeta(s)$ for each $s \in [\gamma, \gamma + 1]$, providing the poles all lie below γ. According to [67, Thm 3.2], this implies that the spectra and multiplicities converge pointwise in a suitable sense. The converse situation is examined here using the sphere and torus as concrete examples. In these examples, the spectra converge pointwise and this implies that the zeta functions converge pointwise in s. However, this is not quite good enough to show that σ converges to 0, as this needs uniform convergence in s.

Lemma 2. *Let $\zeta \neq 0$. If $\zeta_n \to \zeta$ as $n \to \infty$ uniformly in $[\gamma, \gamma + c]$ for any constant $c > 0$, then $\sigma(\zeta_n, \zeta) \to 0$.*

Proof. Suppose $|\zeta_n(s) - \zeta(s)| < \epsilon$ for all $n > n_0$ and all $s \in [\gamma, \gamma + c]$. Then

$$\left|\log \frac{\zeta_n}{\zeta}\right| \leq \left|\frac{\zeta - \zeta_n}{\zeta}\right| \leq \frac{\epsilon}{\zeta}. \tag{185}$$

Hence $\sigma(\zeta_n, \zeta) \leq \epsilon \sup_s 1/\zeta(s) = \epsilon / \min_s \zeta(s)$. Hence the distance σ converges to zero. □

The fuzzy spheres define a sequence of Dirac operators D_n, with $n = N$, the matrix size. This is compared to the Dirac operator D on the spin bundle of S^2 tensored with $\mathbb{C}^2$, so that the multiplicities are doubled. Then the D_n have the same spectrum as D but with a cut-off. Therefore $\zeta_n(s) \to \zeta(s)$ as $n \to \infty$ for $s > 1$ and this is uniform in $[\gamma, \gamma + 1]$ for any $\gamma > 1$. Therefore $\sigma(D_n, D) \to 0$. This conclusion generalises to any sequence of fuzzy spaces for which the spectra are obtained by truncating the spectrum of the limiting geometry.

Now let D_n be a sequence of fuzzy unit square tori with $n = N$, and D the continuum torus, again with multiplicities doubled. Here, the spectra do not coincide but converge pointwise, when ordered in increasing value. The fact that the zeta functions converge uniformly for $\gamma > 1$ is now shown.

For the fuzzy unit square torus, the spin structure is $\Sigma_c = (1,1)$. Therefore the same spin structure is used for the continuum torus and its eigenvalues are labelled with $(k,l) \in A_\infty = \mathbb{Z}^2 + (1/2, 1/2)$. Let $A_N \subset A_\infty$ be the subset such that $-N/2 \leq k,l < N/2$. This indexes the eigenvalues of the fuzzy torus of size N exactly once each, ignoring the fourfold degeneracy of the squared eigenvalues for each (k,l) (see equation eq. (114)). The difference in the two zetas is

$$\Delta\zeta = \sum_{(k,l)\in A_N} ([k]^2 + [l]^2)^{-s} - \sum_{(k,l)\in A_\infty} (k^2 + l^2)^{-s}, \tag{186}$$

where $[k]$ is the q-number introduced in the introduction. All that is left to prove is that $\Delta\zeta \to 0$ as $N \to \infty$, uniformly on $[\gamma, \gamma+1]$, for any $\gamma > 1$.

To show this convergence, the comparison is split into two regions. The fuzzy zeta spectrum is similar in value to the continuum torus for small $[k], [l]$ but for large $[k], [l]$ the spectra differ quite drastically. This can be seen in fig. 3. Define $A_M \subset A_N$ to be the region with $|k|, |l| < M$ for some M such that $1 \leq M \leq N/2$. Then different comparisons are used for A_M and $A_N \setminus A_M$. The difference in the two zetas is thus

$$\Delta\zeta = \sum_{A_M} ([k]^2 + [l]^2)^{-s} - (k^2 + l^2)^{-s} + \sum_{A_N \setminus A_M} ([k]^2 + [l]^2)^{-s} - \sum_{A_\infty \setminus A_M} (k^2 + l^2)^{-s} \tag{187}$$

These three sums are investigated in turn. For the first term,

$$([k]^2 + [l]^2)^{-s} - (k^2 + l^2)^{-s} = \left(\frac{\sin^2 x + \sin^2 y}{\sin^2 z}\right)^{-s} - \left(\frac{x^2 + y^2}{z^2}\right)^{-s} \tag{188}$$

with $x = \pi k/N$, $y = \pi l/N$, $z = \pi/N$. The Taylor expansion for this is

$$\left(\frac{\sin^2 x + \sin^2 y}{\sin^2 z}\right)^{-s} = \left(\frac{x^2 + y^2}{z^2}\right)^{-s} \left(1 + s\big(O(x^2 + y^2) + O(z^2)\big)\right) \tag{189}$$

providing x, y and z are in a sufficiently small region. This requires that M/N be sufficiently small.

Putting this estimate into the sum gives

$$\sum_{A_M}([k]^2+[l]^2)^{-s}-(k^2+l^2)^{-s}\leq s\cdot\sum_{A_M}O\left(\frac{(k^2+l^2)^{1-s}}{N^2}\right)\leq s\,O(M^2/N^2). \tag{190}$$

The second inequality follows from the fact that $s>1$ and so the summand is greatest for $k,l=\pm 1/2$. This shows that as long as $M/N\to 0$, the first piece of expression (187) converges uniformly to zero.

The second term in eq. (187) is examined now. Let $\mathrm{sinc}(x)=(\sin x)/x\leq 1$. The term is

$$\sum_{A_N\setminus A_M}([k]^2+[l]^2)^{-s}\leq N^2\left(\frac{\sin\frac{\pi M}{N}}{\sin\frac{\pi}{N}}\right)^{-2s}=N^2M^{-2s}\left(\frac{\mathrm{sinc}\frac{\pi M}{N}}{\mathrm{sinc}\frac{\pi}{N}}\right)^{-2s} \tag{191}$$

$$\leq N^2M^{-2\gamma}\frac{\left(\mathrm{sinc}\frac{\pi M}{N}\right)^{-2(\gamma+1)}}{\left(\mathrm{sinc}\frac{\pi}{N}\right)^{-2\gamma}} \tag{192}$$

Thus as long as M is chosen so that $NM^{-\gamma}\to 0$ as $N\to\infty$ as well as the previous condition $M/N\to 0$, the bound on the right converges to zero. Hence this sum also converges uniformly to zero.

The last piece is the one arising from the continuum zeta. This converges to zero as long as $M\to\infty$ since it is the tail of a convergent series. The convergence is uniform since

$$\sum_{A_\infty\setminus A_M}\left(k^2+l^2\right)^{-s}\leq\sum_{A_\infty\setminus A_M}(k^2+l^2)^{-\gamma}, \tag{193}$$

a bound independent of s. All of the conditions on M can be satisfied by taking $M=(N/2)^a$ for $1/\gamma<a<1$.

Given the above convergence for the fuzzy sphere and fuzzy torus, some numerical results are presented in fig. 26, where $\gamma=1.5$ to avoid numerical instabilities close to the singularity at $s=d/2=1$. The left-hand plot shows the results for the

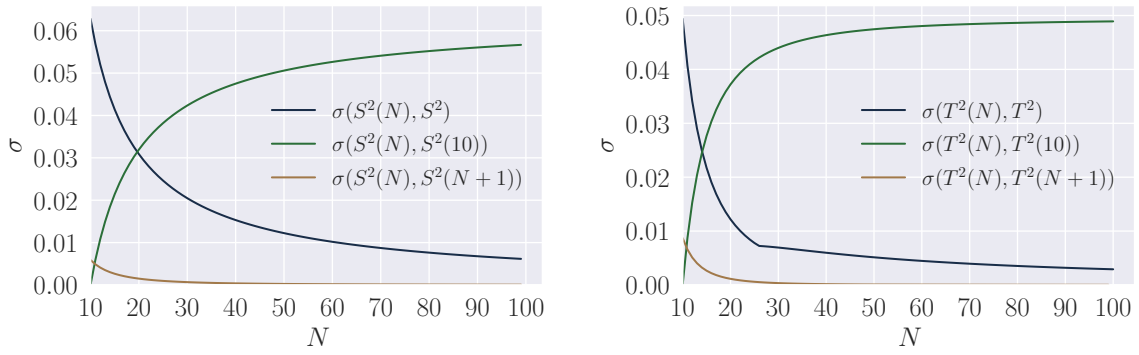


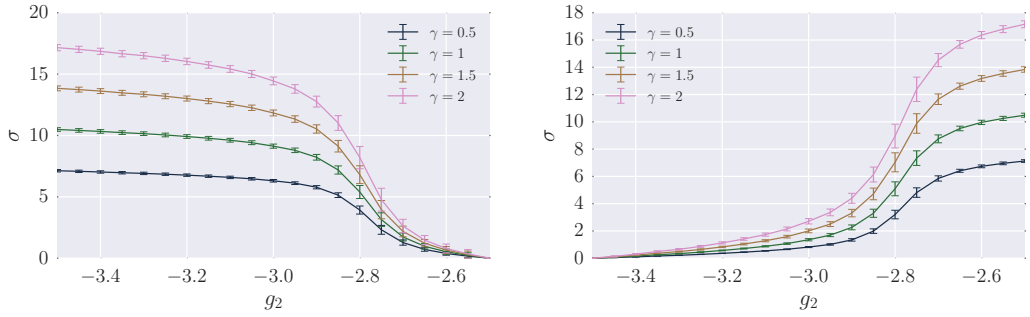
Figure 26.: Spectral distances between fuzzy spaces of different matrix size N and the continuum space, using $\gamma = 1.5$. The sphere is on the left and the torus on the right. The dark blue line shows that the distance of the fuzzy space to the continuum space. The green line denotes the distance of the fuzzy space at $N = 10$ to the fuzzy space at arbitrary N. The brown line shows the distance between the spaces at N and $N + 1$.

fuzzy sphere and the right-hand plot for the fuzzy torus. The plot for the torus has a noticeable kink in the line comparing the continuum torus to the fuzzy torus at $N = 26$. This feature arises because at this point the maximum of the logarithm flips from the upper end of the interval $[\gamma, \gamma + 1]$ to the lower end of this interval.

3.5.2 *Distances between random geometries as g_2 varies*

To apply the distance function to random geometries, the first step is to pick a value of γ. Since the random geometries do not have a simple dimension measure or any poles in the ζ function, the easiest choice is to explore the distance for a few values of γ that would be compatible with the range of dimension found using the spectral variance. For the exploration here $\gamma = 0.5, 1.0, 1.5, 2.0$ are used, which are suitable considering the dimension values found in fig. 21. The choice of γ influences the relative weight given to the low or high energy part of the spectrum, with lower γ emphasising the higher energy part of the spectrum.

Calculating this on the random fuzzy spaces requires some form of averaging, either over distances or over geometries. Calculating averages over quantities in-



(a) Distance from $g_2 = -2.5$ (b) Distance from $g_2 = -3.5$

Figure 27.: Distance of random geometries of type $(2, 0)$ with varying g_2 from the geometries at $g_2 = -2.5, -3.5$

volving the zeta function is plagued by instabilities which arise from terms λ_0^{-s} if the eigenvalue λ_0 with the smallest absolute value fluctuates close to 0. This makes it more practical to calculate the distances of the averaged spectra. Using this, the distance function can be applied to random geometries, obtaining some interesting results.

A simple test is to calculate the distance between geometries of the same type at different g_2. This is shown in fig. 27 for type $(2, 0)$, where the distance from the geometries with $g_2 = -2.5$ and $g_2 = -3.5$ as reference points is shown. The choice of γ changes the resulting distances but the relative distances are qualitatively the same. In particular, the geometries above the phase transition at $g_2 = -2.8$ are more similar to the geometry at $g_2 = -2.5$ while those below the phase transition are more similar to the geometry at $g_2 = -3.5$. The error bars on all distance measures are calculated by propagation of uncertainty starting from the errors on the average eigenvalues.

3.5.3 *Measuring the distance from the fuzzy S^2*

The spectral distance measure can be used to investigate whether the spectra of the random geometries are close to the fuzzy sphere. As demonstrated in [67], the

spectral distance measure is also sensitive to size differences between geometries. However for the random fuzzy spaces their size is of less interest than the question of whether they resemble a fuzzy sphere of any size. Thus to compare the geometries, the average spectra are scaled so that the maximum eigenvalues are equal. Physically this rescaling should correspond to fixing the Planck scale for the geometries to agree.

The resulting distances between the random geometries and the appropriate fuzzy spheres are shown in fig. 28. These are encouraging, since they align with the hope that the geometries close to the phase transition are similar to the fuzzy sphere. For all three types the minimum distance to the fuzzy sphere is for the geometries with g_2 one step away from the phase transition. For type $(1,1)$ it is for $g > g_c$, and for types $(2,0)$ and $(1,3)$ it is for $g < g_c$.

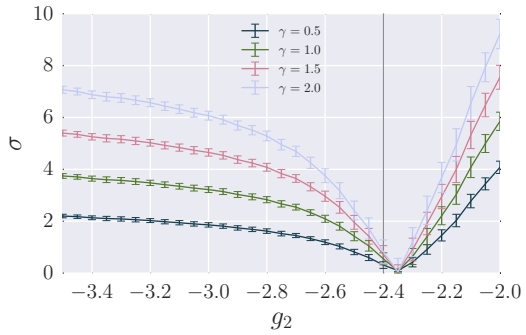
3.5.4 *Measuring the distance between type* $(1,1)$ *and type* $(2,0)$

One of the aims in doing this additional analysis was to understand the difference between the geometries of type $(1,1)$ and $(2,0)$ better. For this a distance measure is very useful. When comparing geometries of different types, in addition to choosing the rescaling, there is freedom in which g_2 values to compare with each other.

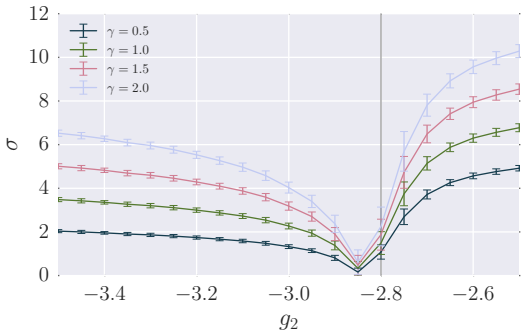
The simplest option is to compare the geometries at the same value of g_2, disregarding the fact that the geometries have different phase transition points. The difference found in this way is dominated by whether the geometries are in the same phase or not, as seen in fig. 29 (a). The distances measured do not change much when the eigenvalues are rescaled to $\lambda_{\max} = 1$. This is because the maximum eigenvalues in both geometries are very similar.

When comparing at fixed g_2 value, the difference between the geometries is largest between $g_2 = -2.8, \ldots, -2.5$ which is the region in which type $(1,1)$ is already transitioned, while type $(2,0)$ is not yet. After $g_2 = -2.8$, when type $(2,0)$ transitions, the distance between the geometries becomes much smaller.

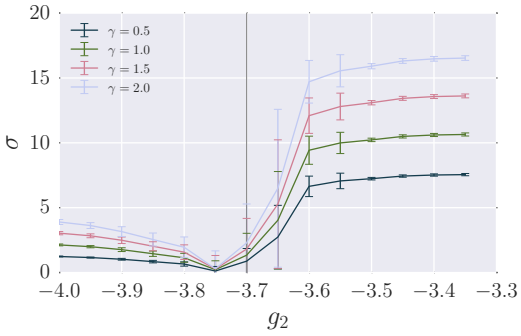
The other option is to, is to compare the geometries at a constant distance from their phase transition $g_2 - g_c$, shown in fig. 29 (b). This requires the assumption that the determination of the phase transition for both geometries is correct, which introduces an additional possible error. The overall distance between the geometries in this measure is smaller than before. It is also much larger when comparing the geometries before they have transitioned as opposed to comparing those in that that are after the phase transition The geometries after the phase transition have distance close to 0, while before the phase transition, their distance is quite large.



(a) Type $(1,1)$ $N = 10$

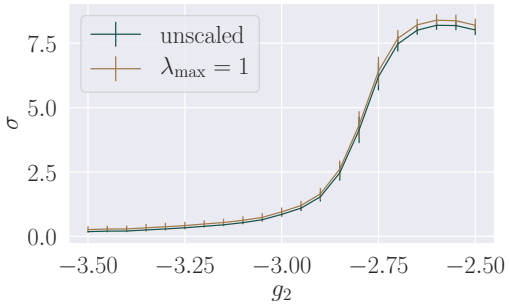


(b) Type $(2,0)$ $N = 10$

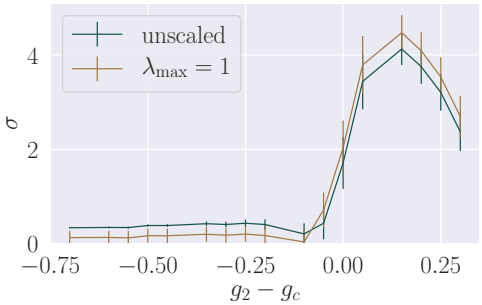


(c) Type $(1,3)$ $N = 8$

Figure 28.: Distance between the fuzzy sphere and random fuzzy geometries, with the spectra rescaled such that their maximal eigenvalues agree. For the geometries of types $(1,1)$ and $(2,0)$ only half of the multiplicity of the fuzzy sphere eigenvalues were used to compare to geometries with the same number of eigenvalues.



(a) At the same g_2



(b) At the same $g_2 - g_c$

Figure 29.: The spectral distance between geometries of type $(1,1)$ and type $(2,0)$ with different rescalings. Plotted for $\gamma = 1$.

4

SYMMETRIES, COADJOINT ORBITS AND FUZZY SPACES

This section is concerned with investigating the construction of fuzzy spaces that possess Lie group symmetries. The fuzzy sphere is a key motivating example as it possesses an action of the Lie group $SO(3)$. The fact that the fuzzy sphere possesses a Lie group symmetry whilst having a finite Dirac spectrum is an attractive feature for a quantum gravity theory. Identifying the maximum eigenvalue of the Dirac operator as the maximum possible energy scale of the geometry, is akin to the existence of a minimum Planck length scale. Such an interpretation is expected to regularise the divergences that appear in the quantum field theories one could define over such spaces. As having a spacetime that possesses a Lorentz invariant metric is a fundamental principle of general relativity, exploring the fuzzy spaces that are invariant under Lie group symmetry seems a useful avenue of investigation.

The sphere is a particularly special space as it has many different descriptions in differential geometry, with varying levels of structure. However the focus of this thesis will be to view the sphere as a coadjoint orbit of the Lie group $SU(2)$. This provides the sphere many geometric structures, such as a canonical symplectic structure, a Kähler structure, and allows it to be viewed a special type of homogeneous space which makes its symmetry explicit. It also allows us to describe it in a Lie algebraic manner, which lends itself to the noncommutative geometry setting. As fuzzy spaces are generalisations of spin manifolds, we need to outline how a manifold possessing a Lie group symmetry, manifests on the spinor bundle. The details of how the Lie group symmetry interacts with the Dirac operator is then given.

The aim of this chapter is to investigate how much the presence of a Lie group symmetry restricts the possible choices of Dirac operators on a fuzzy space as given by eq. (79) and eq. (80).

4.1 DIFFERENTIAL GEOMETRY OF COADJOINT ORBITS

In this section we introduce the necessary differential geometry to define a spinor bundle over a coadjoint orbit. But first we need to describe a coadjoint orbit.

4.1.1 *Coadjoint Orbits of Semisimple Lie Groups*

Definition 34. Let G be a Lie group and let $\mathfrak{g}$ be it's Lie algebra, we say that $\mathfrak{g}$ is semisimple if it does not contain any non-abelian, non-trivial ideals. If G is a Lie group with semisimple Lie algebra we say that G is also semisimple.

Definition 35. A torus subgroup of a Lie algebra is a connected, compact, abelian subgroup. A maximal torus, T, is a torus proper subgroup that satisfies the standard maximal property that there are no proper subgroups, T' of G that contain T with $T \neq T'$.

Definition 36. Let G denote a compact semisimple Lie group, take the Lie algebra of G to be $\mathfrak{g}$. A Cartan subalgebra of $\mathfrak{g}$ is the Lie algebra of a maximal torus subgroup, T, of G.

A Cartan subalgebra can be viewed as the maximal set of commuting elements of $\mathfrak{g}$. We will denote the vector space dual of $\mathfrak{g}$ as $\mathfrak{g}^*$, which we will refer to as the dual Lie algebra (even though it isn't a Lie algebra itself).

Definition 37. The *adjoint action of a Lie group*, G on itself or it's Lie algebra, $\mathfrak{g}$ is denoted by Ad and is defined as:

$$Ad_g(g') = gg'g^{-1}, \qquad Ad_g(X) = gXg^{-1} \tag{194}$$

for all $g' \in G$ and $X \in \mathfrak{g}$. The *adjoint action of the Lie algebra* of a Lie group on itself is denoted by *ad* and is defined to be:

$$ad_X(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} Ad_{\exp tX} Y = [X, Y] \tag{195}$$

for all $X, Y \in \mathfrak{g}$. The coadjoint action of G on $\mathfrak{g}^*$ is denoted Ad^* and defined:

$$Ad^*_g \alpha(X) := \alpha(Ad_{g^{-1}}(X)) = \alpha(g^{-1}Xg) \tag{196}$$

for every element $\alpha \in \mathfrak{g}^*$ and every $X \in \mathfrak{g}$.

Note that if G is connected we have that $Ad(e^X) = e^{ad(X)}$ for every $X \in \mathfrak{g}$ [122].

Definition 38. The Killing form on a Lie algebra $\mathfrak{g}$ is defined to be $B(X, Y) = Tr(ad_X ad_Y)$ for $X, Y \in \mathfrak{g}$.

Note that when $\mathfrak{g}$ is semisimple then B is non-degenerate and when G is also compact Lie group, then B is negative definite [123]. We can also define $\mathfrak{g}$ to be called compact if its Killing form is negative definite for this reason. It is then useful to define $\mathrm{Kil} = -B$, then we have a positive definite form on a compact semisimple Lie algebra $\mathfrak{g}$. We can then identify $\mathfrak{g}^*$ using this new form Kil to $\mathfrak{g}$ under the following identification: $\tilde{X} = \mathrm{Kil}(X, \cdot)$. The Killing form, B, and therefore the bilinear form Kil satisfy the property that

$$\mathrm{Kil}(ad_X Y, Z) = \mathrm{Kil}([X, Y], Z) = \mathrm{Kil}(Y, [Z, X]) = -\mathrm{Kil}(Y, ad_X Z) \tag{197}$$

Now for every point $\mu_o \in \mathfrak{g}^*$ there exists some $Z_o \in \mathfrak{g}$ such that $\mu_o(X) = \mathrm{Kil}(X, Z_o)$ for all $X \in \mathfrak{g}$. We can then form the orbit under the coadjoint group action defined as

$$Ad^*_g \mu_o(X) := \mu_o(Ad_{g^{-1}} X) = \mathrm{Kil}(Ad_{g^{-1}}(X), Z_o) = \mathrm{Kil}(g^{-1}Xg, Z_o). \tag{198}$$

Denote this orbit by $\mathcal{O}_{\mu_o} = \{\alpha \in \mathfrak{g}^* \mid \exists g \in G \text{ s.t. } Ad^*_g(\mu_o) = \alpha\}$. Using the cyclic nature of the trace we can see that for semisimple Lie algebras we have that the coadjoint orbit of a point $\mu_o \in \mathfrak{g}^*$ is equivalent to the adjoint orbit of it's identified point $Z_o \in \mathfrak{g}$ (via Kil).

Let K denote the stability subgroup of G for the coadjoint action of a point $\mu_o \in \mathfrak{g}^*$, i.e. $K = \{k \in G \mid Ad^*_k \mu_o = \mu_o\}$. The map $x \mapsto Ad^*_x(\mu_o)$ defines a diffeomorphism between G/K and the coadjoint orbit $\mathcal{O}_{\mu_o}$. Using this diffeomorphism, we can equally work with the homogeneous space G/K or the orbit itself.

Definition 39. A manifold M with a left G action is said to be a homogeneous space if the action is transitive, i.e. for every $x, y \in M$ there exists a $g \in G$ such that $g \cdot x = y$.

As the action is transitive, we can take any point $x \in M$ and the orbit space $G \cdot x$ which is equal to M. By the orbit-stabiliser theorem of group theory we have a bijection between G/G_x and $G \cdot x = M$, where G_x is the stabiliser of the point. The following theorem allows us to adapt this to the setting of manifolds.

Theorem 6 ([124])**.** *Let G be a Lie group, K a closed subgroup of G. Let G/K be the space of left cosets with the natural topology. Then G/K has a unique smooth structure. If M has a transitive action of G, then the stabiliser, G_x of any point $p \in M$ is closed and M is diffeomorphic to G/K.*

The projection $\pi : G \to G/K$ which send $g \in G$ to the equivalence class $[gK]$ forms a fibre bundle (see [124]). Viewing the coadjoint orbit as a homogeneous space, we can take a new view of the space. We shall view it as a principal K-bundle, with total space G, fibres K and base space will be G/K.

4.1.2 *Homogeneous Bundles Theory*

Definition 40. A fibre bundle $\pi : P \to B$ with fibres F which has a right action ρ_G of a Lie group G on P is called a principal G-bundle if we have the following:

- The action is free. This means that if $\forall x \in P$ we have that $x \cdot g = x \implies g = e$.
- The group acts transitively on the fibre of the bundle. This means that $\forall p, \tilde{p} \in \pi^{-1}(\{x\}) =: P_x$ then there is some $g \in G$ such that $p = \tilde{p} \cdot g$.
- The local trivialisations, ψ_{U_x} satisfy the following conditions. That $\psi_{U_x}(p) = (\pi(p), \phi_{U_x}(p))$ where $\phi : \pi^{-1}(U_x) \to G$ is G-equivariant, i.e. that $\phi_{U_x}(p) \cdot g = \phi_{U_x}(p \cdot g)$. This can be viewed as requiring the following $(id_{U_x} \times \mu_G) \circ (\psi_x \times id_G) : \pi^{-1}(U_x) \times G \to U \times G$ (where μ_G is the group multiplication) is equivalent to $\psi_U \circ \rho_G : \pi^{-1}(U_x) \times G \to U \times G$. I.e. the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_x) \times G & \xrightarrow{\psi_x \times id_G} & U_x \times G \times G \\ \downarrow{\scriptstyle \rho_G} & & \downarrow{\scriptstyle id_{U_x} \times \mu_G} \\ \pi^{-1}(U_x) & \xrightarrow{\psi_x} & U_x \times G \end{array}$$

We say that two principal G bundles $\pi_1 : P_1 \to M$ and $\pi_2 : P_2 \to M$ are equivalent if there exists a homeomorphism $f : P_1 \to P_2$ that is G equivariant, i.e.

$$\begin{array}{ccc} P_1 & \xrightarrow{\quad f \quad} & P_2 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & M & \end{array}$$

commutes and $f(p \cdot g) = f(p) \cdot g$ for every $g \in G$.

Proposition 5. *Let G be a Lie group and K a closed subgroup, let $\pi : G \to G/K$ denote the smooth surjective projection that sends elements to their equivalence class under the quotient in G/K. Then $\pi : G \to G/K$ is a principal K-bundle.*

Proof.
- *The group, K acts freely on the total space*: Freely acting group means that $\forall x \in G$ we have that $x \cdot k = x$ implies that $k = e_K$. So given that K is a closed subgroup of G, we can include K in G in the standard way, then this property is satisfied by the uniqueness of the identity element, e_G in G. Then as $e_K = e_G$ we have shown this to be true.

- *The group acts transitively on the fibres*: We need to show that $\forall g, \tilde{g} \in G_x = \pi^{-1}(\{[xK]\})$ there is some element $k \in K$ such that $g = \tilde{g} \cdot k$. So as $g \in G_x$ by the definition of an equivalence class, $g = x \cdot k$. Also $\tilde{g} = x \cdot \tilde{k}$, so $x = g \cdot k^{-1} = \tilde{g} \cdot (\tilde{k})^{-1}$. So we have that $g = \tilde{g} \cdot (\tilde{k})^{-1} \cdot k$.

- *We need to show the existence of* $\psi : \pi^{-1}(U) \to U \times K$, $p \mapsto (\pi(p), \phi_U(p))$, *where* ϕ_U *is K-equivariant*: The existence of local trivialisations is shown in [124], which allows us to find local sections for every point in G/K. Let $[xK] \in G/K$, then let U_x be a neighbourhood of $[xK]$ in G/K. Let $s : U_x \to G$ be a local section, then we have that the following map $\psi : U_x \times K \to \pi^{-1}(U_x)$, $(p, k) \mapsto s(p) \cdot k$, is K equivariant, i.e. $\psi(p, k) \cdot g = s(p) \cdot k \cdot g = s(p) \cdot (kg) = \psi(p, kg)$.

□

With this new perspective we can the make a very useful link between the *group level* description and the *lie algebra level*. We do this by taking the differential of the map $\pi \colon G \to G/K$ at the identity element $e \in G$. We shall set $o = \pi(e) = [eK]$. We have that the induced linear map $d\pi_e : T_eG \simeq \mathfrak{g} \to T_o(G/K)$. Which by using the first isomorphism theorem from linear algebra we have an isomorphism between $im(d\pi_e) = T_o(G/K)$[1] and $T_eG/ker(d\pi_e)$.

It is worth explicitly showing that $\ker(d\pi_e) \simeq \mathfrak{k}$ (as vector spaces). We shall do this by showing that elements of $\mathfrak{k} \simeq T_e(K)$ are in $\ker(d\pi_e)$ and then we show that $\ker(d\pi_e) \subset \mathfrak{k}$.

So given $X_e \in T_eK \simeq \mathfrak{k}$ (a vector tangent to K), we can form a map $\alpha : [0, 1] \to K$ such that $\alpha(0) = e$ and $\alpha'(0) = X_e$, i.e. let $\alpha(t) = exp(tX_e)$. We can then view K as a subgroup of G and consider the maps α as curves in G. Then we have a curve $\alpha(t) \in K \subset G$ and $\alpha(0) = e$ thus $\pi(\alpha(t)) = \pi(e) = o$ for all values of t. So a vector tangent to this curve must have that $d\pi_e(X_e) = d(\pi_e \circ \alpha)(0) = d(o) = 0_o$. So we have that $\mathfrak{k} \subset ker(d\pi_e) = \{X_e \in T_eG \mid d\pi_e(X_e) = 0_{\pi(e)}\}$.

1 the map $d\pi_p$ is surjective for any $p \in P$ as it is a necessary condition for the existence of local trivialisations in the bundle structure.

Now we take $X_e \in ker(d\pi_e)$ so we have that $d\pi_e(X_e) = 0_o$. Let us choose a curve in G such that $\alpha(t) \in G$, $\alpha(0) = e$ and $\alpha'(0) = X_e$. Then we have that if $d\pi(X_e) = 0$ then we have that $(\pi \circ \alpha)'(0) = \frac{d}{dt}(\pi(\alpha(t))|_{t=0} = 0$. So we have that $\pi(\alpha(t)) = const$, but $\pi(\alpha(0)) = \pi(e) = o$. So we have that $\pi(\alpha(t)) = o$ for all values of t. So $X_e \in \mathfrak{k}$.

So we have that the tangent space to G/K at the element $o = \pi(e)$ is the vector space $\mathfrak{g}/\mathfrak{k}$

$$T_o(G/K) = im(d\pi_e) \simeq T_eG/ker(d\pi_e) = \mathfrak{g}/\mathfrak{k}. \tag{199}$$

Definition 41. A homogeneous space, $M \cong G/K$ is *reductive* if there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $g = \mathfrak{k} \oplus \mathfrak{m}$ where $Ad_k(\mathfrak{m}) \subset \mathfrak{m}$ for all $k \in K$.

Note that for reductive homogeneous spaces we have that $T_oG/K \cong \mathfrak{m}$.

Let $\tau_g : G/K \to G/K$ denote the diffeomorphism that sends $xK \mapsto gxK$. Note that $\tau_k(o) = o$ for all $k \in K$. Taking the differential of this map at o provides a linear transformation of the tangent space.

Definition 42. The *isotropy representation* of the homogeneous space G/K is the homomorphism:

$$I_o : K \to GL(T_oG/K) \tag{200}$$

defined by $k \mapsto (d\tau_k)_o$, where $o = eK$ the identity coset.

The following proposition explains why the adjoint action plays a vital role in the study of homogeneous spaces.

Proposition 6 ([125])**.** *The isotropy representation of a reductive homogeneous space G/K is equivalent to the adjoint representation of K in $\mathfrak{m}$.*

Proof. The two representations are equivalent if the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{\quad Ad_k \quad} & \mathfrak{m} \\ {\scriptstyle d\pi_e|_{\mathfrak{m}}}\downarrow & & \downarrow{\scriptstyle d\pi_e|_{\mathfrak{m}}} \\ T_o(G/K) & \xrightarrow{I_o(k)=(d\tau_k)_o} & T_o(G/K) \end{array}$$

where the downwards arrows are the canonical isomorphism $T_o(G/K) \cong \mathfrak{m}$ given above, the upper right arrow is the restriction of the adjoint action to K and using the fact that the space is reductive so $Ad_k(\mathfrak{m}) \subset \mathfrak{m}$. We show this by calculating explicitly, given $Y \in \mathfrak{m}$ we have that :

$$((d\pi_e)_o \circ Ad_k)(Y) = (d\pi_e)_o(Ad_k(Y)) = \frac{d}{dt}\big|_{t=0}(\pi \circ \exp(tAd_k(Y))) \tag{201}$$

$$= \frac{d}{dt}\big|_{t=0}(\exp(tAd_k(Y)K)) = \frac{d}{dt}\big|_{t=0}(k\exp(tY)k^{-1}K) \tag{202}$$

$$= (d\tau_k)_o \circ \frac{d}{dt}\big|_{t=0}\exp(tY)K)) = (d\tau_k)_o \circ (d\pi_e)_o(Y) \tag{203}$$

□

Definition 43. Let $M = G/K$ be a homogeneous space, a metric g on M is called left G-invariant if for each $g \in G$ the diffeomorphism $\tau_g : G/K \to G/K$, $\tau_g(xK) = gxK$ is an isometry, i.e. we have that $g_o((d\tau_g)_o(X), (d\tau_g)_o(Y)) = g_o(X, Y)$ for every $X, Y \in T_o(G/K) \cong \mathfrak{m}$.

The following theorem allows us to concern ourselves with specific inner products on the space $\mathfrak{m}$ instead of Riemannian metrics on G/K.

Theorem 7 ([126], p.200)**.** *Let G/K be a reductive homogeneous space with $Ad(K)$-invariant decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, then there is a one-to-one correspondence between:*

- *left G-invariant Riemannian metrics on M*
- *$Ad(K)$-invariant inner products $\langle , \rangle$ on $\mathfrak{m}$*

Remark 6. *There is a similar statement for pseudo-Riemannian metrics, where we relax the inner product to a non-degenerate bilinear form on $\mathfrak{m}$.*

Proposition 7. *Given a reductive homogeneous space G/K with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. The positive definite bilinear form Kil is $Ad(G)$-invariant and it's restriction to $\mathfrak{m}$ is $Ad(K)$ invariant.*

Proof. $\text{Kil}(Ad_k(X), Ad_k(Y)) = \text{Kil}(kXk^{-1}, kYk^{-1}) = Tr(ad_{kXk}ad_{kYk^{-1}})$. Note that $ad_{kXk^{-1}}(Y) = [kXk^{-1}, Y] = kXk^{-1}Y - YkXk^{-1} = k(Xk^{-1}Yk)k^{-1} - k(k^{-1}YkX)k^{-1} = Ad_k \circ ad_X \circ Ad_{k^{-1}}(Y)$. So we have that

$$\text{Kil}(Ad_k(X), Ad_k(Y)) = Tr(Ad_k ad_X Ad_{k^{-1}} Ad_k ad_Y Ad_{k^{-1}}) \tag{204}$$

and as Ad is a group homomorphism and the trace is cyclic we have the desired result that $\text{Kil}(Ad_k(X), Ad_k(Y))) = \text{Kil}(X, Y)$. As we have take a reductive decomposition for $\mathfrak{m}$, we have that the result. □

As the aim is to examine differential operators over coadjoint orbits (viewed as homogeneous spaces) we need to introduce the concept of associated bundles and their sections.

Definition 44. If $\pi : E \to M$ is a principal G-bundle and V is a space with a left action of G, $\rho : G \times V \to V$, then we can form the *associated bundle*, $E \times_\rho V$, which has total space $(E \times V)/\sim$, where $(p, v) \sim (q, w)$ if there exists $g \in G$ such that $(p, v) \cdot g := (p \cdot g, \rho(g^{-1})v) = (q, w)$. If we take V to be a vector space which carries a linear representation of G we obtain an *associated vector bundle*.

Every finite dimensional vector bundle, $\pi : E \to M$ of rank n can be viewed as an associative vector bundle of a principal $GL(n)$-bundle over M [70]. There is a particularly useful result relating the sections of an associated bundle to equivariant maps.

Proposition 8 ([70])**.** *Let $\pi : E \to M$ be a principal G bundle, let V have a left action of G, denoted by ρ, and let $E \times_\rho V$ be the associated bundle. Let $\Gamma(M, E \times_G V)$ be the space of smooth sections of the associated bundle. Let $C^\infty(E, V)^G$ be the space of maps $s : E \to V$ such that $s(p \cdot g) = \rho(g^{-1})s(p)$. There is a natural one-to-one correspondence between $\Gamma(M, E \times_\rho V)$ and $C^\infty(E, V)^G$ given by sending the map $s \in C^\infty(E, V)^G$ to $s_M(x) = [p, s(p)]$, where p is any element of $\pi^{-1}(x)$ and $[p, s(p)]$ is the equivalence class that the element of $(p, s(p)) \in P \times G$ belongs to.*

Proof. First we show that this map is unambiguous, i.e. if we choose some other point $p' \in \pi^{-1}(x)$ then we have that s is mapped to $[p', s(p')]$. As G acts transitively on the fibres we have that $p' = p \cdot g$ for some $g \in G$. So

$$[p', s(p')] = [p \cdot g, s(p \cdot g)] = [p \cdot g, \rho(g^{-1})s(p)] = [p, s(p)] \tag{205}$$

where the last equality is due to the definition of the associated bundle. The map is surjective because if we are given $s_M \in \Gamma(E \times_G V)$, we have that $s_M(x) = (p, v)$ for some $p \in \pi^{-1}(x)$ and $v \in V_x$, we can define s to equal v at p. This is well defined because if $s_M(x) = (q, w)$ instead, we have that $q = p \cdot g$ then we just set $v = \rho(g)w$. We check that this map $s(p \cdot g)g^{-1} \cdot s(p)$. Suppose $s(pg) = \tilde{v}$ so we have that $[(pg, \tilde{v})] \in V_{\pi(pg)} = V_{\pi(p)}$. Thus taking the representative $(pg, \tilde{v}) \cdot g^{-1} = (p, \rho(g)\tilde{v})$ we have that $s(p) = \rho(g)\tilde{v}$ so $\rho(g)^{-1}s(p) = s(pg)$. To show that the map is injective, consider two maps s, s' such that $s \neq s'$. So there exists $p \in E$ such that $s(p) \neq s'(p)$, thus we have that choosing x with $p \in \pi^{-1}(x)$ we have that $s_M(x) \neq s'_M(x)$. □

Definition 45. Let $\pi : E \to M$ be a fibre bundle and let G be a Lie group. We say that the fibre bundle is *G-equivariant* if G acts smoothly on the left of both E and M and we have that $\pi(g \cdot x) = g \cdot \pi(x)$ for all $g \in G$ and all $x \in E$. If $\pi : E \to M$ is a vector bundle we also require that $g\cdot : E_x \to E_{g \cdot x}$ is linear.

Considering the bundle $\pi : G \to G/K$ we have a left G action on both G by left multiplication and on G/K by sending xK to gxK. These actions satisfy $\pi(g \cdot x) = gxK = g \cdot xK = g \cdot \pi(x)$, thus this principal K-bundle is also G-equivariant. Now that given an associated bundle to $\pi : G \to G/K$, denoted $G \times_\rho V \to G/K$, is also G-equivariant with the left action of G on $G \times_\rho V$ being defined as $g' \cdot [g, v] := [g'g, v]$. Thus we have that $\pi(g' \cdot [g, v]) = \pi([g'g, v]) = g'g = g' \cdot \pi([g, v])$.

Definition 46. Let M be a homogeneous space of the Lie group G, so that $M \cong G/K$ for some closed subgroup, K, of G. A *homogeneous fibre bundle* over $M \cong G/K$ is a G-equivariant fibre bundle over M where the action of G is given by left multiplication.

A *homogeneous vector bundle* over M is a G-equivariant vector bundle where the action of an element $g \in G$ is a vector bundle homomorphism

Given a left action of K on a vector space V, denoted by $\rho : K \times V \to V$. We can form the associated vector bundle $G \times_\rho V$ of the principal bundle $\pi : G \to G/K$. We have that all homogeneous vector bundles are isomorphic to an associated bundle to $G \to G/K$ for a finite dimensional representation, (ρ, V) of K [127].

Proposition 9. *Let $\pi : G \to G/K$ be a principal-K bundle over the homogeneous space. If we have a Lie group homomorphism $\tau : K \to P$ for some group P we have an left action $\rho : K \times P \to P$, defined by $\rho : (k, p) \mapsto \tau(k) \cdot p$. We can then form the associated bundle $G \times_\rho P$. This bundle is a principal-P bundle over G/K.*

Proof. The action of P on $G \times_\rho P$ is given by $[g, p] \cdot p' := [g, pp']$. This action is free and as the right multiplication of a group on itself is transitive, this action is transitive on the fibres. Let $s : U_x \to P$ be a local section of a point $xK \in G/K$. Define $\psi(x', p) = s(x') \cdot p$, this map is clearly P-equivariant. □

Example 7. *An important example of a principal bundle is that of the frame bundle of a vector bundle. Let $\pi : E \to M$ be a real vector bundle over a M, with n-dimensional fibres. Define $GL(E)$ to be the space of all invertible linear maps $e_x : \mathbb{R}^n \to E_x = \pi^{-1}(x)$. Then $GL(E)$ is a principal bundle for the group $GL_n(\mathbb{R})$ under the action $(p \cdot g)(v) = p(g \cdot v)$ for where $p : \mathbb{R}^n \to E_x$, $g \in GL_n(\mathbb{R})$ and $v \in \mathbb{R}^n$. We say that $GL(E)$ is the frame bundle for E.*

Note that the above construction works for complex vector bundles with minimal modification. The frame bundle of the tangent bundle $\pi : TM \to M$ is often referred to as *the frame bundle*, however any vector bundle can have a frame bundle to which it is the associated vector bundle. In what follows the tangent frame bundle is referred to as FM.

If an n-dimensional manifold is oriented and possesses a Riemannian metric, the structure group of the tangent frame bundle can be reduced to that of $SO(n)$. So the tangent frame bundle is a principal $SO(n)$ bundle.

Given a G-equivariant bundle, $\pi : E \to M$, we have that the action of the group on the sections of this bundle is given by the formula $(g \cdot s)(x) = g \cdot s(g^{-1}x)$, i.e so the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{g\cdot} & E \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle (g\cdot s)} \\ M & \xrightarrow{g\cdot} & M \end{array}$$

4.1.3 *Differential Operators on Homogeneous Spaces*

Differential operators play a vastly important role in physics. With the symmetries of physical theories manifesting themselves in specific symmetries of key differential operators. The most obvious example is that of Lorentz invariance which all of the standard modern theories of physics requires. In physics, these differential operators are often expressed in local coordinates, however to define them in a geometric and coordinate free way, we need to express them as specific combinations of covariant derivatives on vector bundles.

Definition 47. A covariant derivative on a vector bundle $E \to M$ is a map $\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$ (where $T^*M \to M$ is the cotangent bundle), such that for $s \in \Gamma(E)$ and $f \in C^\infty(M)$ we have that $\nabla(fs) = df \otimes s + f\nabla(s)$. If we take a vector field $X \in \Gamma(TM)$, we define $\nabla_X : \Gamma(E) \to \Gamma(E)$ to be $\iota(X)\nabla$ where ι is the contraction operator and refer to it as the covariant derivative by X.

Given a vector bundle $\pi : E \to M$, we can form the *endomorphism bundle* which has total space $End(E) = \cup_{x \in M} End(E_x)$ where $E_x = \pi^{-1}(x)$. Where $End(E_x)$ are the linear maps from E_x to itself.

We now introduce the formal definition of a differential operator, heuristically a differential operator on a vector bundle is a linear combination of covariant derivative, where the coefficients are fibre preserving maps.

Definition 48. Given a vector bundle $E \to M$, a *differential operator* is a linear map from the sections of the bundle, $\Gamma(E)$, to itself, such that it is composed of sections

$a_i \in \Gamma(End(E))$ and covariants derivatives ∇_X where X ranges over all vector fields on M. A differential operator of order m is an element of

$$\mathcal{D}_m(M,E) = \Gamma(End(E)) \cdot \text{span}\{\nabla_{X_1} \cdots \nabla_{X_j} \mid j \leq m\}. \tag{206}$$

Let the space of Differential operators on the vector bundle E be denoted by $\mathbb{D}(E)$.

Definition 49. Given a homogeneous vector bundle $\pi : E \to G/K$, a differential operator on E is called equivariant if $g \cdot Ds = D(g \cdot s)$.

Recall that $Ds \in \Gamma(E)$ and the action of G on sections is given by $(g \cdot s)(x) = g \cdot s(g^{-1}x)$. So for a differential operator to be G-equivariance, we require the following diagram to commute:

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{g\cdot} & \Gamma(E) \\ \downarrow D & & \downarrow D \\ \Gamma(E) & \xrightarrow{g\cdot} & \Gamma(E) \end{array}$$

Note that this definition doesn't explicitly realise which homogeneous space of the Lie group G we are examining. One way to make this more explicit is to move from the pictures of sections of the vector bundles to K-equivariant functions.

As every homogeneous vector bundle is isomorphic to an associated vector bundle of $\pi : G \to G/K$ [127]. Let $\pi : G \times_\rho V \to G/K$ be the associated vector bundle to the map $\rho : K \times V \to V$ and let $C^\infty(G,V)^K$ denote the smooth functions, $f \in C^\infty(G,V)$ that obey $f(gk) = \rho(k^{-1})f(g)$ for all $k \in K$. Let the map that associates sections $s \in \Gamma(G \times_\rho V)$ to an element of $C^\infty(G,V)^K$ be denoted by $\widetilde{s}$ and the reverse map that sends $f \in C^\infty(G,V)^K$ to its associated section be denoted by f^0. Given a differential operator $D \in \mathbb{D}(G \times_\rho V)$, we then define an associated operator on $C(G,V)^K$ in the following way: $\widetilde{D}f = \widetilde{(Df^0)}$. We will refer to this operator $\widetilde{D}$ as a differential operator also. Note that as D is a linear map, we have that $\widetilde{D}$ is also linear on $C^\infty(G,V)^K$. We have that the left action of the Lie group on the algebra of functions is given by $L_g(f)(x) = g \cdot f(x) = f(g^{-1}x)$. So we have that the G-equivariance

condition for differential operators then becomes $L_g(\widetilde{D}) = \widetilde{D}$ where $L_g(\widetilde{D})f(x) := \widetilde{D}(f \circ L_g)(g^{-1}x)$.

There is a Lie algebraic description of these differential operators. Given that element $X \in \mathfrak{g}$ act on G as left-invariant vector fields via the formula

$$Xf(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp(tX)), \tag{207}$$

we get that $X \in \mathfrak{g}$ acts on $C^\infty(G, V)$ also in the same manner. This action of $\mathfrak{g}$ on $C^\infty(G, V)$ induces an action of the universal enveloping algebra, $\mathcal{U}(\mathfrak{g})$, on $C^\infty(G, V)$ also. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is isomorphic to $\mathbb{D}(G \times \mathbb{R})$ viewed as the algebra of operators on $C^\infty(G)$ generated by the left invariant vector fields on G and the identity operator. [124, Ch. II, Proposition 1.9]. So taking $L \in End(V)$ and taking $X \in \mathcal{U}(\mathfrak{g})$ we can set the action of $End(V) \otimes \mathcal{U}(\mathfrak{g})$ on $C^\infty(G, V)$ as $(L \otimes X)f = L(X(f))$. Let K act on $End(V) \otimes \mathcal{U}(\mathfrak{g})$ via the adjoint action in the following way: $\mu(k)(L \otimes X) = \rho(k)L\rho(k^{-1}) \otimes Ad_k(X)$. Let $(End(V) \otimes \mathcal{U}(\mathfrak{g}))^K$ denote the elements that K-invariant under the above action. i.e.

$$(End(V) \otimes \mathcal{U}(\mathfrak{g}))^K = \{D \in End(V) \otimes \mathcal{U}(\mathfrak{g}) \mid \mu(k)(D) = D \ \forall\ k \in K\}. \tag{208}$$

Proposition 10 ([128]). *Let G/K be a reductive homogeneous space with K compact. Then the space of G-equivariant differential operators on $\pi : G \times_\rho V \to G/K$ is bijective with the space $(End(V) \otimes \mathcal{U}(\mathfrak{g}))^K$.*

Note that $1 \otimes D \in End(V) \otimes \mathcal{U}(\mathfrak{g})$ descends to a homogeneous differential operator if $Ad_k(D) = D$ (as $\rho(k)\rho(k)^{-1} = 1$). So for a differential operator on the smooth functions of G to descend to a G-invariant differential operator on G/K, we require that $Ad_k(D) = D$. As the space of differential operators on G is isomorphic to $\mathcal{U}(\mathfrak{g})$, and the universal enveloping algebra is generated by $X \in \mathfrak{g}$ which are view as vector

fields on G given by eq. (207). So for a vector field on G to descend to a differential operator on G/K, we require $Ad_k(X) = X$ for any $k \in K$.

$$Ad_k(X)(f)(x) = X(f \circ Ad_k) \circ Ad_k^{-1}(x) = X(f \circ Ad_k)(k^{-1}xk) \tag{209}$$

$$= \frac{d}{dt}\big|_{t=0} f \circ Ad_k(k^{-1}xk\exp(tX)) = \frac{d}{dt}\big|_{t=0} f(xk\exp(tX)k^{-1}) \tag{210}$$

$$\tag{211}$$

this is equal to $X(f)$ only when $k\exp(tX)k^{-1} = \exp(tX)$. Note that we have that $Ad_k(X) = R_k(X)$ for $X \in \mathfrak{g}$ and therefore this holds on $\mathcal{U}(\mathfrak{g})$, where $R_k(f)(x) = f(xk)$ but $R_k(x) = xk^{-1}$ for $x \in G$. So the elements of $\mathfrak{g}$ that are left G-invariant and right K invariant descend to differential operators on G/K.

We now need to construct the specific differential operator we are keen to investigate, the Dirac operator.

4.1.4 *Spin Structures and Spinor Bundles*

A given Clifford algebra, $Cl(V,Q)$, can be decomposed into the eigenspaces of the map $\alpha : Cl(V,Q) \to Cl(V,Q)$ which extends the map $\alpha(v) = -v$ for $v \in V$ to all of $Cl(V,Q)$, i.e. $\alpha(v_1 \dots v_k) = (-1)^k v_1 \dots v_k$. Note that $\alpha^2 = Id$ so it has eigenvalues ± 1 on $Cl(V,Q)$. Let $Cl^0(V,Q)$ denote the $+1$ eigenspace and $Cl^1(V,Q)$ denote the -1 eigenspace, referred to as the even and odd parts of the Clifford algebra respectively. We have $Cl(V,Q) = Cl^0(V,Q) \oplus Cl^1(V,Q)$.

Definition 50. Let $v \in V \subset Cl(V,Q)$ be such that $Q(v) \neq 0$, let $P(V,Q) \subset Cl(V,Q)$ be the space generated by such $v \in V$. We then define the $Pin(V,Q)$ group of a Clifford algebra to be the subgroup of $P(V,Q)$ generated by elements $v \in V$ $Q(v) = \pm 1$. The $Spin(V,Q)$ group is then the even subgroup of $Pin(V,Q)$, i.e.

$$Spin(V,Q) = Pin(V,Q) \cap Cl^0(V,Q) \tag{212}$$

The spin group is a Lie group and is a double cover of the special orthogonal group $SO(V, Q)$ [62]. We let $Spin(n)$ denote the spin group when $V = \mathbb{R}^n$ and Q is the usual Euclidean inner product.

Given that the spin group is a Lie group, we can consider principal bundles over manifolds that have a spin group as their structure group. We are focussed on a specific $Spin$ bundle over an oriented Riemannian manifold:

Definition 51. Given the tangent frame bundle, $\pi : FM \to M$ of a connected n-dimensional oriented manifold M with a Riemannian metric we can define a *spin structure* on the manifold is a pair (P, Λ) such that:

a) P is a $Spin(n)$-principal bundle over M, $\tilde{\pi} : P \to M$.

b) $\Lambda\colon P \to FM$ is an equivariant 2-fold covering, i.e. the diagram

$$\begin{array}{ccc} P \times Spin(n) & \longrightarrow & P \\ \downarrow{\scriptstyle \Lambda\times\lambda} & & \downarrow{\scriptstyle \Lambda} \\ FM \times SO(n) & \longrightarrow & FM \end{array} \qquad \begin{array}{l} \tilde{\pi} \searrow \\ \qquad M \\ \pi \nearrow \end{array}$$

commutes. Where $\lambda : Spin(n) \to SO(n)$ is the covering map and the horizontal arrows are the right action in the principal bundle structure.

Definition 52. Given a representation of the spin group, $\rho : Spin(n) \to End(V)$ for a vector space V. Let M be a manifold with a spin structure $\pi : P \to M$, then we can form the associated vector bundle $P \times_\rho V$. This is called a *spinor bundle*, sections of this bundle are called spinor fields and they can be viewed as functions $s : P \to V$ such that $s(pg) = \rho(g^{-1})s(p)$ for every $p \in P$ and $g \in Spin(n)$.

Note that this allows for different types of spinors depending on the representation chosen. There in general exist multiple representations of the spin group to choose from, however the consequences of the different choices is not investigated here, see [94] for further details. Given that representations of the spin group are given by restricting an irreducible representation of a Clifford algebra, we can take

V to be a Clifford module, with the Clifford multiplication map $c : Cl(V, Q) \to End(V)$.

4.1.5 *The Dirac Operator*

The covering map Λ allows us to lift various structures from the tangent frame bundle to the spin bundle. The object we require an understanding of is the spin connection.

Definition 53. A connection 1-form, ω, on a principal G bundle $\pi : P \to M$ is a $\mathfrak{g}$ valued map from the tangent bundle TP of the total space, such that it has the following properties:

- $\omega(\tilde{X}) = X$ for each $X \in \mathfrak{g}$ where $\tilde{X}$ denotes X acting as a vector field on P.
- $(R_g)^*\omega = Ad_{g^{-1}}\omega$, i.e. the following diagram commutes:

$$\begin{array}{ccc} T_pP & \xrightarrow{\omega_p} & \mathfrak{g} \\ {\scriptstyle dR_g}\downarrow & & \downarrow{\scriptstyle Ad_{g^{-1}}} \\ T_{pg}P & \xrightarrow[\omega_{pg}]{} & \mathfrak{g} \end{array}$$

Given a vector bundle $\pi : V \to M$, we have the following result:

Proposition 11 ([70], Propositive 1.16)**.** *There is a one-to-one correspondence between connections of the frame bundle $GL(V)$ and covariant derivatives on the vector bundle V.*

Restricting ourselves to considering the tangent frame bundle over a oriented Riemannian manifold, we have that a connection 1-form takes values in $\mathfrak{so}(n)$. Given the covering map $\lambda : Spin(n) \to SO(n)$ we can pull back the connection 1-form from FM to P. Taking the unique Levi-Civita connection (i.e. the metric compatible and torsion free covariant derivative on the tangent bundle) on an oriented Riemannian manifold we can take the associated connection 1-form on the tangent frame bundle. Taking the pullback to the Spin bundle, we get a unique connection one-form. Using this we can form covariant derivatives on the associated vector bundles $P \times_\rho V$

in the following way. View sections of $P \times_\rho V$ as functions $\psi : P \to V$ such that $\psi(p \cdot g) = \rho(g^{-1})\psi(p)$, then define

$$D^\omega \psi = d\psi + \rho_*(\omega)\psi \tag{213}$$

where ρ_* is the differential of the representation $\rho : Spin(n) \times V \to V$ and $d\psi$ is the exterior derivative.

Definition 54. The spinor covariant derivative ∇^S on the bundle $S = P \times_\rho V$ is the map $\nabla^S : \Gamma(S) \to \Gamma(T^*M \otimes S)$ given by the unique lift of the Levi-Civita covariant derivative, ∇, on the tangent bundle TM by the procedure described above.

Taking the cotangent bundle[2] of a Riemannian manifold T^*M, we can form a Clifford algebra at each point by taking $Cl(T^*_xM, g)$ at each point $x \in M$. By the inclusion of $T^*_xM \subset Cl(T^*_xM, g)$ we can use the Clifford multiplication between an element in T^*_xM and a Clifford module (S, c) for $Cl(T^*_xM, g)$. This can be done at every point $x \in M$ giving us an extension of Clifford multiplication to a map from $\Gamma(T^*M \otimes S)$ to $\Gamma(S)$.

So by combining the Clifford multiplication on sections with the spinor covariant derivative ∇^S we get an operator from $\Gamma(S)$ into itself.

Definition 55. Given a Riemannian spin manifold, M, with a spinor bundle S, the *Dirac operator* is given by

$$\not{D}\psi = (c \circ \nabla^S)\psi \tag{214}$$

where $\psi \in \Gamma(S)$.

4.1.6 *Spin Geometry on Homogeneous Spaces*

Recall that the isotropy representation for a reductive homogeneous space, G/K is a map $I_o : K \to GL(\mathfrak{m})$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the reductive decomposition. When

2 By making use of the so-called musical isomorphisms between the tangent and cotangent bundles we could equally take the tangent bundle as was done in section 2.1.

this homogeneous space is an oriented Riemannian reductive homogeneous space the isotropy representation maps into $SO(\mathfrak{m})$. We saw in prop. 6 that the isotropy representation is equivalent to the adjoint representation.

The following theorem allows us to assert the existence of spin structures on homogeneous Riemannian manifolds in terms of the existence of lifts of the isotropy representation.

Theorem 8 ([129]). *Let M be an n-dimensional, oriented, connected Riemannian manifold with a transitive Lie group G of orientation-preserving isometries. Let K be the isotropy group of a point o of M and $Ad \cong \tau : K \to S0(n)$ be the linear isotropy representation. Then*

- *If τ lifts to $\tilde{\tau} : K \to Spin(n)$, then there is a spin structure on M such that $P = G \times_{\tilde{\tau}} Spin(n)$*
- *If $\tilde{\tau}$ and $\tilde{\tau}'$ are two lifts of τ and the spin structures defined by P and $P' = G \times_{\tilde{\tau}} Spin(n)$ are isomorphic, then $\tilde{\tau} = \tilde{\tau}'$.*
- *If the group G is simply connected and M has a spin structure, then τ lifts to $Spin(n)$.*

Thus if we can lift the adjoint action $Ad : K \to SO(\mathfrak{m})$ to an action $\widetilde{Ad} : K \to Spin(\mathfrak{m})$ such that the following diagram commutes:

$$\begin{array}{ccc} & & Spin(\mathfrak{m}) \\ & \overset{\widetilde{Ad}}{\nearrow} & \downarrow \lambda \\ K & \xrightarrow{Ad} & SO(\mathfrak{m}) \end{array}$$

then we have a spin structure defined over G/K. The question of existence of these lifts is non-trivial and is not investigated here.

Given the group homomorphism $Ad : K \to SO(n)$ and $\widetilde{Ad} : K \to Spin(n)$, we can form the associated principal $SO(n)$ ($Spin(n)$) bundle to $\pi : G \to G/K$ by following prop. 9.

Proposition 12 ([130]). *The tangent frame bundle FM over an oriented connected Riemannian homogeneous manifold is equivalent to the associated principal $SO(n)$ bundle $G \times_{Ad} SO(n)$.*

Similarly we have that the principal Spin(n) bundle is the same as $G \times_{\widetilde{Ad}} Spin(\mathfrak{m})$. Given this, we have that the spinor bundles associated to the map $\rho : Spin(n) \to End(V)$, which is a restriction of the Clifford module (c, V), can be expressed in the following terms:

Proposition 13 ([130]). *Given a spin structure $P = G \times_{\widetilde{Ad}} Spin(\mathfrak{m})$ over a connected oriented Riemannian homogeneous manifold G/K. The spinor bundle $P \times_\rho V$ is the same as $G \times_{\rho \circ \widetilde{Ad}} V$.*

Given that sections of an associated bundle can be viewed as equivariant functions, we have that spinors $\psi \in \Gamma(P \times_\rho V)$ are $\rho \circ \widetilde{Ad}(K)$-equivariant functions from G to the Clifford module V, i.e. $\psi(k) = \rho(\widetilde{Ad}(g^{-1}))\psi(g)$.

So using the results of section 4.1.3 along with the knowledge that spinors are sections of the associated bundle $G \times_{\rho \circ \widetilde{Ad}} V$ for some irreducible Clifford module (V, c) such that $c|_{Spin(n)} = \rho$. We can describe the G-equivariant differential operators on the spinor bundle of a homogeneous space G/K, as being elements of $D \in End(V) \otimes \mathcal{U}(\mathfrak{g})$ that obey the following: let $D = \sum \omega^i \otimes X_i$ for $\omega^i \in End(V)$ and $X_i \in \mathcal{U}(\mathfrak{g})$ then they must satisfy:

$$\mu(k)(D) := \sum_i (\rho \circ \widetilde{Ad})(k)\omega^i(\rho \circ \widetilde{Ad})(k^{-1}) \otimes Ad_k(X_i) = D \tag{215}$$

for all $k \in K$.

Example 8. *The sphere is a homogeneous space, $SU(2) \to SU(2)/U(1)$. Thus we have a $SU(2)$-equivariant principal bundle given by the Hopf fibration. Given that $Spin(2) \cong U(1) \cong SO(2)$ we naturally have a lift of any representation of $U(1)$ into $Spin(2)$. Also we have that Dirac operator is an $SU(2)$-equivariant differential operator - see [113, Section 9.A].*

The example of the sphere can be suitably generalised. By the work of Rieffel in [131, 132] it is shown that for a coadjoint orbit of a compact Lie group, the metric induced by the form *Kil* is part of a G-invariant Kähler structure. It is also shown that the Levi-Civita connection for this metric is G-invariant. It is also shown that the

Dirac operator for this connection is G-equivariant [131, Prop 5.9]. This motivates the study of G-equivariant Dirac operators as a potential pathway to constructing fuzzy spaces that approximate manifolds with symmetry. However, the precise way to implement the K-invariance mentioned above is not so clear at this point. However the expression of differential operators on homogeneous spaces as being elements of $(End(V) \otimes \mathcal{U}(\mathfrak{g}))^K$ seems very reminiscent of the Dirac operator for a fuzzy space being an element of $End(V) \otimes End(M_n(\mathbb{C}))$. Investigating this potential link is likely a fruitful avenue of research.

4.2 TRANSFORMATIONS OF FUZZY SPACES

We now turn out attention to the action of Lie groups on fuzzy spaces. Specifically we investigate what restrictions are imposed by requiring the Dirac operators to be equivariant with respects to a Lie group.

Definition 56. A general transformation of a fermion space $(\mathcal{A}, \mathcal{H}, s, \Gamma, J)$ is a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that

- For every $a \in \mathcal{A}$ there exists some $b \in \mathcal{A}$ such that $U\rho(a)U^{-1} = \rho(b)$.
- $U\Gamma = \Gamma U$
- $UJ = JU$

A transformation then sends a Dirac operator D into another Dirac operator D' with the same spectrum:

$$D' = UDU^{-1}. \tag{216}$$

Thus transformations are invariants of the spectral geometry of fuzzy spaces. Any transformation that leaves the Dirac operator the same is called a *symmetry*, i.e. $D' = D$. And given a transformation U, any such Dirac operator such that $D' = D$, i.e. $DU = UD$ will be called equivariant with respects to U.

Proposition 14 ([46]). *A transformation of a fuzzy space acts on an element $v \otimes m \in \mathcal{H}$ as:*

$$U(v \otimes m) = hv \otimes \rho(g) m \rho(g)^{-1} \tag{217}$$

for some unitary element $g \in \mathcal{A}$ and a unitary $h : V \to V$ that satisfies $h\gamma = \gamma h$. Any such g, h in turn defines a transformation.

Thus applying a transformation to a Dirac operator as defined in eq. (75) we get that $\omega_i \mapsto h\omega_i h^{-1}$ and $K_i \mapsto \rho(g) K_i \rho(g^{-1})$.

If we restrict ourselves to the case that $q - p \bmod 8 \neq 1, 5$ then the Dirac operator is expressible as follows:

$$D = \sum_i \alpha^i \otimes [L_i, \cdot] + \sum_j \beta^j \otimes \{H_j, \cdot\}, \tag{218}$$

where we have that α_i, β_j are products of an odd numbers of gamma matrices, and that α^i, L_i are anti-Hermitian and that β^j, H_j are Hermitian. The action of a transformation on such a Dirac operator gives us (dropping the notation ρ):

$$\begin{aligned} UDU^*(v \otimes m) &= UD(h^* v \otimes g^* m g) \\ &= U(\sum_i \alpha^i h^* v \otimes [L_i, g^* m g] + \sum_j \beta^j h^* v \otimes \{H_j, g^* m g\}) \\ &= \sum_i h\alpha^i h^* v \otimes g\,[L_i, g^* m g]\, g^* + \sum_j h\beta^j h^* v \otimes g\,\{H_j, g^* m g\}\, g^*) \end{aligned}$$

Note that we have:

$$g\,[L_i, g^* m g]\, g^* = [g L_i g^*, m] \tag{219}$$

$$g\,\{H_j, g^* m g\}\, g^* = \{g H_j g^*, m\} \tag{220}$$

so we get that

$$UDU^* = \sum_i h\alpha^i h^* \otimes [gL_i g^*, \cdot] + \sum_j h\beta^j h^* \otimes \{gH_j g^*, \cdot\} \tag{221}$$

We now specify to specific Clifford types to investigate equivariant Dirac operators.

4.2.1 *Type* $(1,3)$ $SU(2)$*-equivariant Dirac operators*

For a type $(1,3)$ fuzzy space we have that $V = \mathbb{C}^4$ and within the Dirac operator we have that the Hermitian terms are:

$$h\gamma^0 h^* v \otimes \{gH_0 g^*, m\}, \qquad h\gamma^1\gamma^2\gamma^3 h^* v \otimes \{gH_{123} g^*, m\} \tag{222}$$

and that the anti-Hermitian terms are:

$$h\gamma^i h^* v \otimes [gL_i g^*, m] \qquad \sum_{i<j} h\gamma^0\gamma^i\gamma^j h^* v \otimes [gL_{ij} g^*, m]. \tag{223}$$

So for a $SU(2)$ action on this fuzzy space we need a transformation for every $b \in SU(2)$ provided by:

$$U(b)\colon \mathbb{C}^4 \otimes M_m(\mathbb{C}) \to \mathbb{C}^4 \otimes M_m(\mathbb{C}), \quad U(b) = \psi(b) \otimes I_{\rho_m(b)}, \tag{224}$$

where $I_{\rho_m(b)}$ is the conjugate action of the m-dimensional representation of b, $\rho_m(b)$, and ψ is some representation of $SU(2)$ in $\mathbb{C}^4$ that obeys the constraints in prop. 14. So we have that

$$\begin{aligned} \psi\colon SU(2) &\longrightarrow End(V) \\ b &\longmapsto h = \psi(b) \end{aligned}$$

and

$$\rho_m \colon SU(2) \longrightarrow End(M_m(\mathbb{C}))$$
$$b \longmapsto g = \rho_m(b)$$

If we pass to the infinitesimal picture by letting $b = 1 + \epsilon\tilde{b}$, with $\tilde{b} \in \mathfrak{su}(2)$, then we have that that $h = 1 + \epsilon\tilde{h}$ and $g = 1 + \epsilon\tilde{g}$, where $\tilde{h}$ and $\tilde{g}$ are in the derived representations $\psi_*(\mathfrak{su}(2))$ and $(\rho_m)_*(\mathfrak{su}(2))$ respectively. We note that we take $\psi_*(\mathfrak{su}(2))$ to map the generators T_i to the spin representation of $\mathfrak{su}(n)$, which are given by $\psi_*(T_i) = -\frac{1}{2}f_{ijk}S^{jk} = -\frac{1}{8}f_{ijk}[\gamma^j, \gamma^k] = -\frac{1}{4}f_{ijk}\gamma^j\gamma^k$, where $S^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$. Note that γ^i and γ^j are the anti-Hermitian generators of the (1,3)-Clifford module and the γ^i anti-commute with the element γ^0 (i.e. $\gamma^i\gamma^0 = -\gamma^0\gamma^i$) Thus we arrive at the following expression for the action of $SU(2)$:

$$h\gamma^0 h^* v \otimes \{gH_0g^*, m\} = (1 + \epsilon\tilde{h})\gamma^0(1 - \epsilon\tilde{h})v \otimes \{(1 + \epsilon\tilde{g})H_0(1 - \epsilon\tilde{g}), m\} \tag{225}$$
$$= \gamma^0 v \otimes \{H_0, m\} + \epsilon\left([\tilde{h}, \gamma^0]v \otimes \{H_0, m\} + \gamma^0 v \otimes \{[\tilde{g}, H_0], m\}\right) + o(\epsilon^2) \tag{226}$$

So we need the ϵ term to vanish for this operator to be equivariant. For the $[\tilde{h}, \gamma^0]$ term, we examine this by expanding out the formula:

$$\begin{aligned}
[\tilde{h}, \gamma^0] &= -\frac{1}{8}h^i f_{ijk}[[\gamma^j, \gamma^k], \gamma^0] = -\frac{1}{4}h^i f_{ijk}[\gamma^i\gamma^j, \gamma^0] \\
&= -\frac{1}{4}h^i f_{ijk}(\gamma^j\gamma^k\gamma^0 - \gamma^0\gamma^j\gamma^k) = -\frac{1}{4}h^i f_{ijk}(\gamma^j(-\gamma^0\gamma^k) - \gamma^0\gamma^j\gamma^k) \\
&= -\frac{1}{4}h^i f_{ijk}(-(-\gamma^0\gamma^j)\gamma^k - \gamma^0\gamma^j\gamma^k) = 0
\end{aligned}$$

So in order for the ϵ term to vanish we need that $[\tilde{g}, H_0]$ vanishes for any element $\tilde{g} \in (\rho_m)_*(\mathfrak{su}(2))$. This requires that H_0 be in the center of $\mathfrak{su}(2)$, which is equal to multiples of the identity. Examining the term $h\gamma^1\gamma^2\gamma^3 h^* v \otimes \{gH_{123}g^*, m\}$ in a similar manner we have that we need:

$$[\tilde{h}, \gamma^1\gamma^2\gamma^3]v \otimes \{H_{123}, m\} + \gamma^1\gamma^2\gamma^3 v \otimes \{[\tilde{g}, H_{123}], m\} = 0 \tag{227}$$

For anti-Hermitian Clifford generators we have that:

$$[\gamma^i\gamma^j, \gamma^a] = -2\delta^{aj}\gamma^i + 2\delta^{ai}\gamma^j \tag{228}$$

$$[\gamma^i\gamma^j, \gamma^a\gamma^b] = -2\delta^{aj}\gamma^i\gamma^b + 2\delta^{ai}\gamma^j\gamma^b - 2\delta^{bj}\gamma^a\gamma^i + 2\delta^{bi}\gamma^a\gamma^j \tag{229}$$

$$[\gamma^i\gamma^j, \gamma^a\gamma^b\gamma^c] = (-2\delta^{aj}\gamma^i + 2\delta^{ai}\gamma^j)\gamma^b\gamma^c \tag{230}$$

$$+ \gamma^a(-2\delta^{bj}\gamma^i + 2\delta^{bi}\gamma^j)\gamma^c \tag{231}$$

$$+ \gamma^a\gamma^b(-2\delta^{cj}\gamma^i + 2\delta^{ci}\gamma^j) \tag{232}$$

$$[\gamma^i\gamma^j, \gamma^{a_1}\gamma^{a_2}\cdots\gamma^{a_n}] = (-2\delta^{a_1 j}\gamma^i + 2\delta^{a_1 i}\gamma^j)\gamma^{a_2}\cdots\gamma^{a_n} \tag{233}$$

$$+ \gamma^{a_1}(-2\delta^{a_2 j}\gamma^i + 2\delta^{a_2 i}\gamma^j)\gamma^{a_3}\cdots\gamma^{a_n} \tag{234}$$

$$+ \cdots + \gamma^{a_1}\cdots\gamma^{a_{n-1}}(-2\delta^{a_n j}\gamma^i + 2\delta^{a_n i}\gamma^j) \tag{235}$$

$$-\frac{1}{4}f_{ijk}[\gamma^j\gamma^k, \gamma^{a_1}\gamma^{a_2}\cdots\gamma^{a_n}] = f_{ija_1}\gamma^j\gamma^{a_2}\cdots\gamma^{a_n} + f_{ija_2}\gamma^{a_1}\gamma^j\gamma^{a_3}\cdots\gamma^{a_n} \tag{236}$$

$$+ \cdots + f_{ija_n}\gamma^{a_1}\gamma^{a_2}\cdots\gamma^{a_{n-1}}\gamma^j \tag{237}$$

So for the case above we have that (note that here we have specified to 3 anti-Hermitian gamma matrices):

$$[\tilde{h}, \gamma^1\gamma^2\gamma^3] = -\frac{1}{4}h^i f_{ijk}[\gamma^j\gamma^k, \gamma^1\gamma^2\gamma^3] \tag{238}$$

$$= h^i(f_{ij1}\gamma^j\gamma^2\gamma^3 + f_{ij2}\gamma^1\gamma^j\gamma^3 + f_{ij3}\gamma^1\gamma^2\gamma^j) \tag{239}$$

$$= h^i(f_{i11}\gamma^1\gamma^2\gamma^3 + f_{i21}\gamma^2\gamma^2\gamma^3 + f_{i31}\gamma^3\gamma^2\gamma^3 \tag{240}$$

$$+ f_{i12}\gamma^1\gamma^1\gamma^3 + f_{i22}\gamma^1\gamma^2\gamma^3 + f_{i32}\gamma^1\gamma^3\gamma^3 \tag{241}$$

$$+ f_{i13}\gamma^1\gamma^2\gamma^1 + f_{i23}\gamma^1\gamma^2\gamma^2 + f_{i33}\gamma^1\gamma^2\gamma^3 \tag{242}$$

$$= h^i(-f_{i21}\gamma^3 + f_{i31}\gamma^2 - f_{i12}\gamma^3 - f_{i32}\gamma^1 \tag{243}$$

$$+ \quad f_{i13}\gamma^2 - f_{i23}\gamma^1) \tag{244}$$

$$= h^i(-\gamma^1(f_{i23} + f_{i32}) + \gamma^2(f_{i13} + f_{i31}) - \gamma^3(f_{i21} + f_{i12}) \tag{245}$$

$$= 0. \tag{246}$$

Where the last equality comes from the total antisymmetry of the structure constants f_{ijk}. So we require that $[\tilde{g}, H_{123}] = 0$ also. Which again means that H_{123} is an element of the center of $\mathfrak{su}(2)$ i.e. is proportional to the identity element.

Examining the anti-Hermitian term $h\gamma^i h^* v \otimes [gL_i g^*, m]$ we have that we need the following to vanish

$$[\tilde{h}, \gamma_i]v \otimes [L_i, m] + \gamma^i v \otimes [[\tilde{g}, L_i], m] \tag{247}$$

Again as we have that $\tilde{h} = -\frac{1}{2}h^m f_{mjk} S^{jk}$, we can use the rules from above to have that:

$$[\tilde{h}, \gamma^i] = -\frac{1}{4}h^m f_{mjk}[\gamma^j\gamma^k, \gamma^i] = -\frac{1}{4}h^m f_{mjk}(-2\delta^{ik}\gamma^j + 2\delta^{ij}\gamma^k) \tag{248}$$

$$= \frac{1}{2}h^m(f_{mji}\gamma^j - f_{mik}\gamma^k) = h^m f_{mji}\gamma^j \tag{249}$$

So, we need that $\gamma^i v \otimes [[\tilde{g}, L_i], m] = -h^m f_{mji}\gamma^j \otimes [L_i, m]$. As $\tilde{g} \in (\rho_m)_*(\mathfrak{su}(2))$, we need that $[\tilde{g}, L^i] = g^a[T_a, L_i] = h^a f_{aij} L_j$, where indices have been relabelled for clarity. Note that $[T_a, T_b] = f_{abc}T_c$. A solution is to take $L_i = T_i$ and $g^a = h^a$, so we have that $g^a[T_a, T_i] = g^a f_{aij}T_j = -g^a f_{aji}T_j$ as needed. More clearly

$$[\tilde{h}, \gamma_i]v \otimes [L_i, m] + \gamma^i v \otimes [[\tilde{g}, L_i], m] = h^m f_{mji}\gamma^j v \otimes [L_i, m] + g^a f_{aij}\gamma^i v \otimes [L_j, m] \tag{250}$$

$$= f_{mij}\gamma^i v \otimes [L_j, m](g^m - h^m) = 0 \tag{251}$$

Note that the above calculation does not explicitly use the fact that we are dealing with 3 anti-Hermitian gamma matrices and holds for any number of anti-Hermitian gamma matrices.

The other anti-Hermitian term, $\sum_{i<j} h\gamma^0\gamma^i\gamma^j h^* v \otimes [gL_{ij}g^*, m]$, yields the following term that needs to vanish:

$$\sum_{i<j}[\tilde{h}, \gamma^0\gamma^i\gamma^j]v \otimes [L_{ij}, m] + \gamma^0\gamma^i\gamma^j \otimes [[\tilde{g}, L_{ij}], m] \tag{252}$$

We have that

$$[\tilde{h}, \gamma^0\gamma^i\gamma^j] = -\frac{1}{4}h^a f_{abc}[\gamma^b\gamma^c, \gamma^0\gamma^i\gamma^j] \tag{253}$$

$$= -\frac{1}{4}h^a f_{abc}(\underbrace{[\gamma^b\gamma^c, \gamma^0]}_{=0}\gamma^i\gamma^j + \gamma^0[\gamma^b\gamma^c, \gamma^i\gamma^j]) \tag{254}$$

$$= -\frac{1}{4}h^a\gamma^0 f_{abc}\left(-2\delta^{ci}\gamma^b\gamma^j + 2\delta^{bi}\gamma^c\gamma^j - 2\delta^{cj}\gamma^i\gamma^b + 2\delta^{bj}\gamma^i\gamma^c\right) \tag{255}$$

$$= \frac{1}{2}h^a\gamma^0(f_{abi}\gamma^b\gamma^j - f_{aic}\gamma^c\gamma^j + f_{abj}\gamma^i\gamma^b - f_{ajc}\gamma^i\gamma^c) \tag{256}$$

$$= \frac{1}{2}h^a\gamma^0(2f_{abi}\gamma^b\gamma^j + 2f_{abj}\gamma^i\gamma^b) = h^a f_{abi}\gamma^0\gamma^b\gamma^j + h^a f_{abj}\gamma^0\gamma^i\gamma^b \tag{257}$$

So we have that the first term in eq. (252) becomes

$$[\tilde{h}, \gamma^0\gamma^i\gamma^j]v \otimes [L_{ij}, m] = h^a(f_{abi}\gamma^0\gamma^b\gamma^j v \otimes [L_{ij}, m] + f_{abj}\gamma^0\gamma^i\gamma^b v \otimes [L_{ij}, m]) \tag{258}$$

So we need that the term $\gamma^0\gamma^i\gamma^j v \otimes [[\tilde{g}, L_{ij}], m]$ exactly cancels these contributions. This suggests L_{ij} is a product of the $\mathfrak{su}(2)$ generators, as we have that

$$[T_a, T_iT_j] = [T_a, T_i]T_j + T_i[T_a, T_j] = f_{aib}T_bT_j + f_{ajb}T_iT_b. \tag{259}$$

However, T_iT_j is not anti-Hermitian. As $(T_iT_j)^\dagger = T_j^\dagger L_i^\dagger = L_jL_i = L_iL_j + f_{jik}L_k \neq -L_iL_j$, so taking $L_{ij} = T_iT_j$ is not an option. However, we can take $L_{ij} = f_{ijk}T_k$, which is anti-Hermitian as the structure constants are real valued. This is akin to taking $L_{ij} = [T_i, T_j]$. We also need to restrict to summing $i < j$ because $T_{ij} = -T_{ji}$ and $T_{ii} = 0$, so if we summed over all i and j, we would cancel everything. Note that this is assumed from the beginning anyways as $\gamma^i\gamma^j = -\gamma^j\gamma^i + 2\eta^{ij}$, so to remove this complication we always order the indices.

Armed with this information we can reduce the formula in eq. (258) by using the following property of the structure constants for $\mathfrak{su}(n)$ which arises due to the Jacobi identity that the Lie bracket satisfies:

$$[T_a, [T_i, T_j]] = [[T_a, T_i], T_j] + [T_i, [T_a, T_j]] \tag{260}$$

$$f_{ijb}[T_a, T_b] = f_{aib}[T_b, T_j] + f_{ajb}[T_i, T_b] \tag{261}$$

$$f_{ijb}f_{abk}T_k = f_{aib}f_{bjk}T_k + f_{ajb}f_{ibk}T_k \tag{262}$$

$$(f_{ijb}f_{abk} - f_{aib}f_{bjk} - f_{ajb}f_{ibk})T_k = 0 \tag{263}$$

$$f_{aib}f_{bjk} + f_{ijb}f_{bak} + f_{ajb}f_{ibk} = 0 \tag{264}$$

This allows us to assert the following, (where indices have been relabelled from above):

$$[f_{aib}L_{bj} + f_{ajb}L_{ib}, m] = [f_{aib}f_{bjk}T_k + f_{ajb}f_{ibk}T_k, m] \tag{265}$$

$$= [-f_{ijb}f_{bak}T_k, m] \tag{266}$$

$$= [f_{ijb}f_{abk}T_k, m] \tag{267}$$

$$\tag{268}$$

So we need that $\sum_{i<j}[\tilde{g}, L_{ij}] = -\sum_{i<j} h^a f_{ijb}f_{abk}T_k$,

$$\sum_{i<j}[\tilde{g}, L_{ij}] = \sum_{i<j}[\tilde{g}, f_{ijb}T_b] = \sum_{i<j} g^a[T_a, f_{ijb}T_b] = \sum_{i<j} g^a f_{ijb}f_{abk}T_k \tag{269}$$

So these terms only vanish if we have that $g^a = -h^a$. Which leads us to the following conclusion. We cannot have both of the anti-Hermitian terms present if we require $SU2$ equivariance.

Again note that the number of anti-Hermitian gamma matrices is not used, so this derivation is valid for any number.

So summarising our results here, we have that the Hermitian matrices must commute with all elements of the Lie algebra $\mathfrak{su}(2)$ and we can only have one of the two

anti-Hermitian terms. So the two forms for the $SU(2)$ equivariant Dirac operators are the following

$$D = \begin{cases} \alpha\gamma^0 \otimes 1 + \beta\gamma^1\gamma^2\gamma^3 \otimes 1 + \delta\gamma^i \otimes [L_i, \cdot] \\ a\gamma^0 \otimes 1 + b\gamma^1\gamma^2\gamma^3 \otimes 1 + c\sum_{i<j} \gamma^0\gamma^i\gamma^j \otimes [f_{ijk}L_k, \cdot] \end{cases} \tag{270}$$

We can notice that $\gamma^1\gamma^2\gamma^3 = i\gamma^0(-i\gamma^0\gamma^1\gamma^2\gamma^3)$, and that $\gamma^i = \gamma^0\gamma^0\gamma^i$. So we can rewrite these operators as:

$$D = \begin{cases} \gamma^0(\alpha \otimes 1 + i\beta\gamma \otimes 1 + \delta\gamma^0\gamma^i \otimes [L_i, \cdot]) \\ \gamma^0(a \otimes 1 + ib\gamma \otimes 1 + c\sum_{i<j} \gamma^i\gamma^j \otimes [f_{ijk}L_k, \cdot]) \end{cases} \tag{271}$$

Expressing in terms of a $(0,3)$ *Clifford Module*

We can construct the (1,3) geometry from smaller Clifford modules by taking suitable *products* of Clifford modules as described in section 2.2.4. To construct the $(1,3)$ Clifford module from a $(0,3)$ Clifford module we need to product it with a type $(1,0)$.

We can take the gamma matrices to be:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^a = \begin{pmatrix} 0 & i\sigma^a \\ -i\sigma^a & 0 \end{pmatrix} \tag{272}$$

where σ^a are the generators of a $(0,3)$ Clifford module acting on $\mathbb{C}^2$ with real structure. With this expression of the gamma matrices we can write the Dirac operators as:

$$D = \left\{ \begin{pmatrix} 0 & \alpha \otimes 1 + i\beta\sigma^1\sigma^2\sigma^3 \otimes 1 + i\delta\sigma^i \otimes [L_i, \cdot] \\ \alpha \otimes 1 - i\beta\sigma^1\sigma^2\sigma^3 \otimes 1 - i\delta\sigma^i \otimes [L_i, \cdot] & \end{pmatrix} \right. \\ \left. \begin{pmatrix} 0 & a + ib\sigma^1\sigma^2\sigma^3 \otimes 1 + \sum_{j<k} cf_{ijk}\sigma^i\sigma^k \otimes [L_k, \cdot] \\ a - ib\sigma^1\sigma^2\sigma^3 \otimes 1 + \sum_{j<k} cf_{ijk}\sigma^i\sigma^k \otimes [L_k, \cdot] & 0 \end{pmatrix} \right\} \tag{273}$$

If we let $\sigma^a = (-i\tilde{\sigma}^a)$, where $\tilde{\sigma}^a$ are the Pauli matrices. Then we have that $i\sigma^1\sigma^2\sigma^3 = -\tilde{\sigma}^1\tilde{\sigma}^2\tilde{\sigma}^3 = -i\mathbb{I}_{2\times 2}$. Using this we have that $ib\sigma^1\sigma^2\sigma^3 \otimes 1 = -ib\mathbb{I}_2$. So we can rewrite the expression above slightly more compactly as:

$$D = \left\{ \begin{pmatrix} 0 & (\alpha - i\beta) \otimes 1 + i\delta\sigma^i \otimes [L_i, \cdot] \\ (\alpha + i\beta) \otimes 1 - i\delta\sigma^i \otimes [L_i, \cdot] & \end{pmatrix} \right. \\ \left. \begin{pmatrix} 0 & (a - ib) \otimes 1 + \sum_{j<k} cf_{ijk}\sigma^i\sigma^k \otimes [L_k, \cdot] \\ (a + ib) \otimes 1 + \sum_{j<k} cf_{ijk}\sigma^i\sigma^k \otimes [L_k, \cdot] & 0 \end{pmatrix} \right\} \tag{274}$$

Chiral Rotations

The operator $R = \exp(i\alpha\Gamma)$ is a chirality rotation operator, a transformation we can perform on the finite noncommutative geometry in the same way as the $SU(2)$ action above. For even values of $s = q - p \bmod 8$, we take $R = \exp(-i\pi\Gamma/4)$. The modified

Dirac operator $\tilde{D} = RDR^{-1}$, maybe or may not be another Dirac operator in general, but for even s we have that:

$$\begin{aligned}
\tilde{D} &= (\cos(\pi/4) - i\sin(\pi/4)\Gamma)D(\cos(\pi/4) + i\sin(\pi/4)\Gamma) && (275)\\
&= (\cos(\pi/4) - i\sin(\pi/4)\Gamma)^2 D = \exp(-i\pi\Gamma/4)\exp(-i\pi\Gamma/4)D && (276)\\
&= \exp(-i\pi\Gamma/2)D = \cos(\pi/2) - i\sin(\pi/2)\Gamma D && (277)\\
&= -i\Gamma D && (278)
\end{aligned}$$

where we have use that $\Gamma D = -D\Gamma$. For $s = 2, 6$ we have that $\tilde{D}$ is again a Dirac operator (done by checking the axioms and using the relations between J and Γ).

So for the $(1, 3)$ Dirac operators above, we should examine what $-i\Gamma D$ are for the two $SU(2)$ equivariant operators. Lets look at

$$\begin{aligned}
-i\Gamma D &= -i\Gamma(\alpha\gamma^0 \otimes 1 + \beta\gamma^1\gamma^2\gamma^3 \otimes 1 + \delta\gamma^i \otimes [L_i, \cdot]) && (279)\\
&= -i(\gamma \otimes 1)(\alpha\gamma^0 \otimes 1 + \beta\gamma^1\gamma^2\gamma^3 \otimes 1 + \delta\gamma^i \otimes [L_i, \cdot])) && (280)\\
&= -i(\alpha\gamma\gamma^0 \otimes 1 + \beta\gamma\gamma^1\gamma^2\gamma^3 \otimes 1 + \delta\gamma\gamma^i \otimes [L_i, \cdot]) && (281)
\end{aligned}$$

So we need to examine the products $\gamma\gamma^\mu$ for $\mu = 0, 1, 2, 3$. Starting with $\mu = 3$, we get the following:

$$\begin{aligned}
\gamma\gamma^3 &= i\gamma^0\gamma^1\gamma^2, & \gamma\gamma^2 &= -i\gamma^0\gamma^1\gamma^3, && (282)\\
\gamma\gamma^1 &= i\gamma^0\gamma^2\gamma^3, & \gamma\gamma^0 &= i\gamma^1\gamma^2\gamma^3, && (283)\\
\gamma\gamma^1\gamma^2\gamma^3 &= -i\gamma^0. && && (284)
\end{aligned}$$

So we end up with

$$\begin{aligned}
-i\Gamma D = &- i(\alpha(i\gamma^1\gamma^2\gamma^3) \otimes 1 + \beta(-i\gamma^0) \otimes 1\\
&+ i\gamma^0\delta(\gamma^2\gamma^3 \otimes [L_1, \cdot] - \gamma^1\gamma^3 \otimes [L_2, \cdot] + \gamma^1\gamma^2 \otimes [L_3, \cdot]) && (285)
\end{aligned}$$

This can be summarised by

$$-i\Gamma D = \alpha\gamma^1\gamma^2\gamma^3 \otimes 1 - \beta\gamma^0 \otimes 1 + \delta \sum_{i<j} \epsilon_{ijk}\gamma^0\gamma^i\gamma^j \otimes [L_k, \cdot] \tag{286}$$

As $f_{ijk} = \epsilon_{ijk}$ for $\mathfrak{su}(2)$ we have that this is of the form of the other type of Dirac operator in eq. (270) with $\delta = c$, $\alpha = b$ and $\beta = -a$. So taking account of the chiral transformations, there is only one form of Dirac operator on a type $(1,3)$ fuzzy space, which we will take as:

$$D = \gamma^0(a \otimes 1 + b\gamma \otimes 1 + c \sum_{i<j} \gamma^i\gamma^j \otimes [L_{ij}, \cdot]) \tag{287}$$

where $L_ij = f_{ijk}L_k$. Note there is no relationship between the coefficients a, b, c imposed. So in particular, the fuzzy sphere as described by Barrett is precisely the case when $a = c$ and $b = 0$, is not the unique $SU(2)$ equivariant Dirac operator.

4.2.2 *Type* $(0,3)$ $SU(2)$ *Equivariant Dirac Operators*

For a type $(0,3)$ Clifford module we have that, $s = 3$ and the Hermitian gamma products are $\gamma^1\gamma^2\gamma^3$ and $(\gamma^i)^2 = -1$ (so constants) and the anti-Hermitian products are γ^i and $\gamma^i\gamma^j$ (for $i < j$). So we have the following Hermitian terms in the Dirac operator:

$$h\gamma^1\gamma^2\gamma^3 v \otimes \{gH_{123}g^*, m\} \qquad v \otimes \{gH_{ii}g^*, m\} \tag{288}$$

and the following anti-Hermitian terms:

$$h\gamma^i h^* v \otimes [gL_i g^*, m] \qquad \sum_{i<j} h\gamma^i\gamma^j h^* v \otimes [gL_{ij}g^*, m] \tag{289}$$

By following a similar procedure to the $(1,3)$ case we arrive at the following forms for the $SU(2)$ equivariant Dirac operators:

$$D = \begin{cases} \alpha \otimes 1 + \delta\gamma^i \otimes [L_i, \cdot] \\ \alpha \otimes 1 + \lambda\gamma^i\gamma^j \otimes [f_{ijk}L_k, \cdot] \end{cases} \tag{290}$$

Again we notice that the Grosse-Prešnadjer Dirac operator is given when $\alpha = \delta$ and so is not the unique $SU(2)$-equivariant Dirac operator for $(0,3)$ Clifford modules.

4.2.3 *Extracting the metric*

For boty type $(1,3)$ and type $(0,3)$ fuzzy spaces the $SU(2)$-equivariance is not enough to select the fuzzy sphere Dirac operator given by Barrett or Grosse-Prešnadjer.

To examine the metric content, we note that the metric tensor is contained in the symbol of squared Dirac operator D^2. That is the coefficients in front of the second order terms must equal the metric. The lower order terms are governed by the connection which may not be Levi-Civita. As the second order terms will arise from the double commutator terms in the square of the fuzzy Dirac operators D^2, we can extract the vielbeins under the following identification: that $[L_{ij}, \cdot] \to X_{ij} := x_i\partial_j - x_j\partial_i$, which both satisfy the same commutation relations. Focusing on the $(0,3)$, as both the $(1,3)$ and $(0,3)$ have the same commutator terms, we have that:

$$D = \alpha \otimes 1 + \lambda \sum_{i<j} \gamma^i\gamma^j \otimes [L_{ij}, \cdot] \tag{291}$$

for $i < j$.

Using the identification $[L_{ij}, \cdot] \to X_{ij}$ and using that that for $(0,3)$ we can take $\gamma^j = i\sigma^j$, where σ^j are the usual Pauli matrices we have that: $\gamma^i\gamma^j = -i\epsilon^{ijk}\gamma^k$. We then find that:

$$D = \alpha \otimes 1 - i\lambda\sigma^3 \otimes X_{12} + i\lambda\sigma^2 \otimes X_{13} - i\lambda\sigma^1 \otimes X_{23} \tag{292}$$

Expanding this and gathering each derivative term yields:

$$D = a - i\lambda(\sigma^2 x^3 - \sigma^3 x^2)\partial_1 - i\lambda(\sigma^3 x^1 - \sigma^1 x^3)\partial_2 - i\lambda(\sigma^1 x^2 - \sigma^2 x^1)\partial_3 \tag{293}$$

Then comparing this to the expression: $D = ie^{\alpha a}\sigma_a \nabla_\alpha$ we can read off the entries for the inverse veilbein. Setting $\lambda = 1$ to simplify matters we arrive at:

$$(e^{\alpha a}) = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix} \tag{294}$$

This inverse veilbein gives the inverse metric of:

$$g^{-1}(x^1, x^2, x^3) = \begin{pmatrix} (x^2)^2 + (x^3)^2 & -x^1 x^2 & -x^1 x^3 \\ -x^1 x^2 & (x^1)^2 + (x^3)^2 & -x^2 x^3 \\ -x^1 x^3 & -x^2 x^3 & (x^1)^2 + (x^2)^2 \end{pmatrix} \tag{295}$$

which is precisely the metric given in eq. (85), which was shown to be the inverse to the round metric on the sphere in section 2.3.1.

So the $SU(2)$-equivariant Dirac operators for type $(1,3)$ and $(0,3)$ fuzzy geometries all produce the round metric on the sphere in the continuum limit. It would be interesting to analyse the spectrum of one of the arbitrary $SU(2)$-equivariant fuzzy Dirac operators and apply the spectral measures of chapter 3. It should be found that the produce similar results to the fuzzy spheres.

5

CONCLUSION AND DISCUSSION

The aim of this program of research was to better understand the subject of fuzzy spaces. An emphasis was made to develop tools to measure familiar geometric constructs in the context of fuzzy spaces, and attain non-trivial values.

Using two concrete examples of fuzzy spaces that are known to have a manifold as their commutative limit, the fuzzy sphere and fuzzy torus. Several proposals for a dimension and volume measure were examined. The quality of a spectral measure was judged based on what it produced for these cases. With a measure being deemed inappropriate if it produced values that were drastically different from the continuum value.

As the fuzzy tori have drastically different spectra to their continuum counterparts, with the largest magnitude eigenvalues differing the most. Their spectra do not obey the asymptotic expression predicted by Weyl's law. Thus, a naive implementation of Weyl's law would not be appropriate for fuzzy spaces, and an analysis was conducted in chapter 3 to explain why.

Locating the poles of the spectral zeta function (which occur at $s = d/2$ for Laplace-type operators) for fuzzy spaces also was deemed inappropriate. A number of reasons were identified for this, despite a procedure being developed that works well for both the fuzzy sphere and fuzzy torus. Multiple fuzzy spaces of different matrix sizes were required in order for this to be defined. This is an unattractive feature as it is not always possible to ensure the same geometry is being considered - such as when investigating the random fuzzy spaces. The dimension is also an

intrinsic property of a fuzzy space and should be calculable from a single spectrum. Even in the cases when it is effective, the convergence to the continuum value is logarithmic in the matrix size. This makes it useless in general for the small matrix sizes considered in this thesis.

The spectral dimension was adapted to the fuzzy geometry case, where it exhibits an unpleasant feature. Originally it was developed to produce a scale dependent measure of dimension from the scalar Laplacian spectrum, which always contains a zero eigenvalue. When the spectrum of an operator does not contain zero as an eigenvalue (such as the Dirac operator for curved spaces) the spectral dimension, $d_s(t)$ grows linearly as the parameter t is increased. This is not a problem in the continuum setting as the topological dimension is defined as $t \to 0$. However the spectral dimension vanishes at $t = 0$ for fuzzy spaces, as was seen in fig. 11. It is therefore necessary to examine some non zero value of the parameter t, for which the linear growth begins to interfere. At this point a new modification called the spectral variance was defined to remove this linear mode. The spectral variance, $v_s(t)$, again vanishes as $t \to 0$ and it also vanishes as $t \to \infty$. It produces remarkably good results for the fuzzy sphere, with the global maximum attaining a value close to 2 for very small matrix sizes. When applied to the fuzzy torus a extraneous bump for small values of t is seen, that becomes sharpened and shifted towards $t = 0$ as the matrix size is increased. After the bump a plateau in the spectral variance occurs, similar to that seen for the fuzzy sphere and the continuum cases. Thus this bump is a considered a consequence of the fuzzy torus spectrum deviating from a classical spectrum for the largest eigenvalues. The plateau is at a height that is very close to the continuum dimension of 2.

The spectral variance also exhibits an interesting feature when applied to a non-square fuzzy torus. When applied to the $a = 3$ fuzzy torus, the spectral variance has two plateaus. With the plateau nearest the origin having a height similar ~ 2 and occurring for similar values of t as the plateau for the square fuzzy torus. The second plateau occurs for larger value of t and has a height of ~ 1. This is interpreted as Kaluza-Klein type effect, where the spectral variance is able to measure

the longer direction of the torus at lower energies (large t), whilst only rendering the full dimensionality of the fuzzy torus for higher energies (small t).

The analysis of the square fuzzy torus required the use of larger matrix sizes than for the fuzzy sphere. This is due to the fuzzy sphere having a Dirac spectrum that is a truncation of the continuum spectrum. Where the fuzzy torus is a dramatically different spectrum, where very few of the eigenvalues match the continuum. It only recovering the continuum values in the infinite matrix size limit.

The non-square fuzzy tori requires even larger matrix sizes than the unit square fuzzy torus. This is because a fuzzy torus with parameters (a, b, c, d) and a matrix size of N, is effectively $ad - bc$ many copies of an (a, b, c, d) fuzzy torus with matrix size $N/(ad - bc)$. As a result the spectrum is that of a $N/(ad - bc)$ torus repeated $ad - bc$ times. As such matrix sizes that are $ad - bc$ larger are required to get qualitatively similar results to a square fuzzy torus of the same matrix size. This is an effect of the cyclicity of the quantum number defined in section 2.3.2 as was shown in fig. 17.

There are a number of ways to get a single number out as the dimension from the spectral dimension and variance, as they are scale dependent dimensional measures. It is clear that it is not necessarily the global maximum of the spectral variance but rather the height of the plateau closest to the origin. This definition is a little vague and further work is required to develop a more rigorous statement. However for the random geometries there is only one extremum which avoids this issue.

When considering the random fuzzy geometries there are two main ways to go about calculating the spectral measures. The first is to calculate the spectral variance for each geometry in the ensemble and average. The second is to take the (increasingly ordered) eigenvalues of each geometry within the ensemble, average the eigenvalues and then apply the spectral variance. A comparison of the two approaches was presented for the type $(2, 0)$ geometries, where it was seen that away from the phase transition both procedures produce very similar results. Near the phase transition however the two methods produce slightly different results. By applying the spectral variance to samples taken from the simulation, it was shown that

all the geometries before and after the phase transition behave qualitatively different. Notably, all the geometries before the phase transition behave similar to each other, and all the geometries after the phase transition also behave like each other. Near the phase transition, the sample contained a mix of geometries from before and after the transition. Which lead to a larger variance in the averaged eigenvalues and therefore a larger variance in the spectral measures.

Despite this the resulting spectral variances are not drastically different with the main differences showing when t is large. The maximum for instance, is minimally changed. The spectral variance also verifies the findings of [63] that the type $(1,1)$ random geometries behave differently when compare with the type $(1,3)$ and $(2,0)$. With the spectral variance undergoing a smooth increase through the phase transition, whereas for the other types the spectral variance experiences a drastic increase in value near the phase transition and remains there afterwards.

An interesting feature was found by examining the maximum of the spectral variance as a function of the action coupling constant g_2 for the different sizes of matrices considered. For the type $(2,0)$ and $(1,3)$ geometries the maximum of the spectral variance appears to be independent of the matrix size exactly at the phase transition. With it decreasing/increasing with the matrix size before/after the phase transition. This has the effect of making the plot of the maximum (fig. 21) have a steeper transition as the matrix size increases yet the value at the phase transition remains ~ 2. This behaviour is unexpected and suggests that the behaviour of the random geometries might be independent of the matrix size at the phase transition. This fact alone provides motivation for further study into the random geometries. Specifically, with larger matrix sizes the behaviour may become clearer.

Two volumes measures were investigated. One based upon the Dixmier trace and another based upon the work by Stern [66]. The Stern procedure is similar to the Dixmier trace with the main advantage of converging faster. The main obstacle for defining a volume measure is that they typically depends on the dimension, and the measures introduced here are not exempt from this. However for the case of the fuzzy sphere and fuzzy torus the dimension is taken to be 2 and both measures

produced respectable results. With the Stern volume converging to the continuum value rapidly. By using the value of the spectral variance $v_s(t)$ as the dimension for varying t, the volume measures were able to made scale-dependent. With the results for the $N = 90$ fuzzy torus being shown in fig. 23. This shows the dimension measures to be closest to the continuum value when the spectral variance is closest to the continuum dimension value at small t.

For the random fuzzy spaces the maximum of the spectral variance is taken as the dimension. The volume measures are then applied with this value for the dimension. Due to the face that the eigenvalues for the random geometries are constrained to lie with an interval $\sim [-3, 3]$ about the origin, the volume increases as the matrix size is increased. This is a result of the eigenvalues being more densely packed instead of attaining greater values. This has the analogous effect in continuum geometry of forcing the points in the space to become further apart, and thus the volume to be larger. Despite this, there is a clear behavioural change in the volume graphs that coincides with the phase transition.

There was an attempt to counteract the action enforce scaling by multiplying the eigenvalues with a Weyl's law factor $N^{1/d}$. This succeeded in removing the N dependent growth of the volume and provided us with fig. 25. It is shown that before the phase transition the geometries were roughly the same size and at the phase transition the volume undergoes an abrupt change. Especially for the type $(2, 0)$ which had the strongest phase transition.

The work of Stern [66] also provides us with a mechanism to access the further poles of the spectral zeta function and therefore further geometrical quantities. It would be intriguing to develop a measure for the second heat kernel coefficient as this is intimately related to the integral of the scalar curvature. Which then could plausibly be used to define a Einstein-Hilbert type action for fuzzy spaces. The higher heat kernel coefficients have a less clear geometric interpretation.

The zeta distance introduced at the end of chapter 3 allows us to quantitatively compare fuzzy spaces to each other and fuzzy spaces to continuum spaces. This confirmed the intuition that the fuzzy spheres become more alike and become closer

(spectrally) to the continuum sphere as the matrix sizes increase. This result was also found for the fuzzy tori. This zeta distance is particularly informative when applied to the random geometries. It is found that the geometries are closest to that of a fuzzy sphere near the phase transition (including for the type $(1,1)$ geometries). This provides a quantitative statement equivalent to the findings of [61]. That the random fuzzy spaces are the most manifold-like near the phase transition.

The final chapter aimed to explore the consequence of a manifold possessing a Lie group symmetry has on the different structures defined upon it. Specifically the aim was to understand how the group action manifests in terms of a symmetry of the Dirac operator. In turn, all G-equivariant differential operators on a homogeneous space G/K were described as elements of $End(V) \otimes \mathcal{U}(\mathfrak{g})$ which are K-invariant. The existence of spin structure on the homogeneous space is described in terms of the existence of a lift of the isotropy representation to the spin group. When such a lift exists, the G-invariant differential operators on a spinor bundle can be classified in a Lie algebraic way. This characterisation strikes a resemblance to the fact that the Dirac operators for a fuzzy space are members of $End(V) \otimes End(M_n(\mathbb{C}))$ and investigating this link is left to future work. The possible $SU(2)$-equivariant Dirac operators on the type $(1,3)$ and $(0,3)$ fuzzy spaces are investigated. They are found to take one of two forms in either case, with the forms in the type $(1,3)$ case being chiral rotations of each other. It was shown that all the Dirac operators found produce the round metric for the sphere, despite the operator being different from the usual Grosse-Prešnadjer or Barrett operators. Thus the G-equivariance does not appear to be enough to select the Levi-Civita connection.

Looking to possible avenues of future research there are some obvious pathways. With regards to the random geometries investigated, the main limitation was that only small matrix sizes were investigated. With the maximum size studied here being 10 by 10 matrices. Despite the fuzzy torus being a particularly special fuzzy space, one that possesses a symmetry, most of the measures defined did not show clear behaviour until around $N = 15$. For a general random geometry, larger matrices are likely to be needed to investigate their properties with confidence.

The work of Stern has allowed for a spectral expression for the second heat kernel coefficient, which is related to the integral of the scalar curvature. Investigating the curvature of fuzzy spaces would be highly interesting and this outlines a direction of study to this effect.

Another avenue of research is to investigate G-equivariant Dirac operators on fuzzy spaces for different Lie groups, G. For instance is it known that $\mathbb{C}P^n$ for odd n are spin manifolds and also coadjoint orbits of $SU(n+1)$. So by investigating the possible $SU(n+1)$-equivariant Dirac operators on fuzzy spaces for odd values of n, may lead to a description of fuzzy $\mathbb{C}P^n$ that satisfies the axioms laid out in section 2.2. For even values of n it is known that $\mathbb{C}P^n$ are not spin manifolds but spin-c [62]. There are also constructions being developed that may incorporate a larger class of geometries such as spin-c geometries. Most of these involve relaxing some of the axioms outlined in section 2.2. The most common approach to allow generalised real-structures that satisfy a modified first order condition. [133, 134, 97, 135, 98, 136, 137]. Thus, there may still be some understanding gained when considering $SU(n)$-equivariant Dirac operators for even n to this effect.

There is also related work, in the direction of twisting the real structure, to move from Euclidean signatures to Lorenzian signatures [138, 139]. A possible avenue of study is to investigate if a characterisation is possible of the finite versions of these so-called *twisted spectral triples*, similar to that provided by Barrett for real spectral triples in [46]. If so this could lend itself to exploration via Monte Carlo methods as was done discussed in this thesis.

In the Lie theoretic description of the fuzzy sphere presented in chapter 4, explicit use of the isotropy group K did not come into play [1]. This is most likely due to their only existing one non-trivial homogeneous space of $SU(2)$. Looking at the case for $SU(3)$, there are multiple different possible homogeneous spaces one can form. Selecting one boils down to choosing a specific choice of isotropy group K. Investigating how to implement the K-invariance conditions in fuzzy geometry is

1 It is used in the definition of the algebra of functions for the continuum case appendix A, but for fuzzy spaces this is not currently understood.

therefore a necessary step in order to define objects such as fuzzy $\mathbb{C}P^n$. A reasonable approach comes from the study of Dirac operators in representation theory. The Dirac operators on homogeneous spaces then correspond to the K-invariant elements of $End(V) \otimes \mathfrak{g}$ for a specific adjoint action of K. This framework seems likely to be adapted to the fuzzy geometry case.

Overall, the aim of this thesis was the explore the world of noncommutative geometry through analysis of fuzzy spaces. Tools were developed to investigate and understand the unusual geometry these spaces possess. It was shown that manifold-like fuzzy spaces can be extracted from a path-integral over all fuzzy spaces. Also, as there exists fuzzy spaces which possess a full Lie group symmetry, such as the fuzzy sphere and fuzzy tori. This suggests that a fuzzy space which is Lorentz invariant could well be possible, once Lorentzian geometries are included into the framework of noncommutative geometry.

All of this promotes the idea that the development of a fuzzy space analogue of the Einstein-Hilbert action could be possible, complete with its full symmetries. Such a development would put noncommutative geometry onto similar lines of investigation as other quantum gravity models, such as LQG, CDT and causal sets. With the standard model already being described by a finite spectral triple, NCG provides a possible framework to investigate the unification of gravity and the rest of physics.

A

ALGEBRA OF FUNCTIONS FOR COADJOINT ORBITS

In this appendix we collate the results necessary to describe the algebra of functions of coadjoint orbits/homogeneous spaces in terms of their representation theory.

Using the identification of coadjoint orbits of compact Lie groups, G, with homogeneous spaces of the form G/K, where K is a closed subgroup (equal to the coadjoint-stability group of a given weight $0 \neq \mu_0 \in \mathfrak{g}^*$). We then view functions on the coadjoint orbit as functions on the homogeneous space G/K. Denote the space of continuous functions on G/K as $C(G/K)$, these can be either real or complex valued (determined by context or explicitly specified). Let $C(G)$ denote the space of continuous functions on the entire Lie group G. Then the algebra $C(G/K)$ can be viewed as a subalgebra of $C(G)$ such that the functions $f \in C(G)$ satisfy $f(xk) = f(x)$ for all $x \in G$ and all $k \in K$, i.e. $C(G/K)$ are the right K-invariant functions on G.

Definition 57. Let G be a compact Lie group, and let (π, V) be a unitary finite dimensional representation of G. Any function on G of the form $f_{u,v}(g) = (\pi(g)u, v)$ for $u, v \in V$ is said to be a matrix coefficients of the representation (π, V). The space of matrix coefficients is denoted by $MC(G)$.

Proposition 15 (Theorem 3.21. Sepanski [123])**.** *The space of matrix coefficients, $MC(G)$, coincides with the space of G-finite vectors of $C(G)$,*

$$C(G)_{G-fin} = \{f \in C(G) \mid dim(span\{l_g(f) \mid g \in G\} < \infty\}, \tag{296}$$

where $l_g(f)(x) = f(g^{-1}x)$ for all $g, x \in G$.

Remark 7. *Note that there is a right action of G on $C(G)$ given by $r_g(f)(x) = f(xg)$, and it turns out that the set $\{f \in C(G) \mid dim(span\{r_g(f) \| \|g \in G\} < \infty\}$ is the same as $C(G)_{G-fin}$, so we can unambiguously talk about* the *G-finite vectors.*

Theorem 9 ([123])**.** *Let G be a compact Lie group. As a $G \times G$-module with $(g_1, g_2) \in G \times G$ acting as $r_{g_1} \circ l_{g_2}$ on $C(G)_{G-fin}$, we have that:*

$$C(G)_{G-fin} \cong \bigoplus_{\pi \in \widehat{G}} E_\pi^* \otimes E_\pi \tag{297}$$

where $\widehat{G}$ is the set of equivalence classes of irreducible (unitary) representations of G and $C(G)_{G-fin} = \{f \in C(G) \mid dim(span\{g \cdot f \mid g \in G\} < \infty\}$

Theorem 10 ([123])**.** *Let G be a compact Lie group. Then $C(G)_{G-fin}$ is dense in both $C(G)$ and $L^2(G)$. (Note that $C(G)$ is also dense in $L^2(G)$ so by the transient property of denseness we it suffices to show that $G(G)_{G-fin}$ is dense in $C(G)$.)*

As we need $A = C^\infty(G)$ and $A = C^\infty(G/K)$, we note that the smooth functions are dense in the (compactly supported) continuous functions and therefore dense in L^2-functions. So we can approximate any function by a sequence of smooth functions.

Corollary 2 ([123])**.** *Let G be a compact Lie group. Considered as a $G \times G$ module with action $r \circ l$ as above, we have that:*

$$L^2(G) \cong \widehat{\bigoplus}_{\pi \in \widehat{G}} E_\pi^* \otimes E_\pi \tag{298}$$

Given that the space of matrix coefficients $MC(G)$ is the same as the space of G-finite vectors of C(G), we have that we can write any continuous function or square-integrable function as a limit of matrix coefficients by the above theorem. Given a unitary irreducible representation of G we have that $f \in C(G)$ can be expressed as

$$f(g) = \sum_{\pi \in \widehat{G}} f^\pi(g) \tag{299}$$

where

$$f^{\pi}(g) = \sum_{ij=2}^{dim(\pi)} c_{ij}^{\pi} f_{v_i v_j}^{\pi}(g), \tag{300}$$

where the v_i and v_j are orthonormal basis elements in the representation (π, V) and c_{ij}^{π} are just coefficients. To clear up the notation we will write:

$$f(g) = \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{dim(\pi)} c_{ij}^{\pi} f_{ij}^{\pi}(g). \tag{301}$$

This is often expressed in another notation using Wigner D-matrices, here we have that $\pi \in \widehat{G}$ is replaced by J labelling irreps and then $D_{mn}^{J}(g) = f_{v_m v_n}^{\pi}(g)$.

The coefficients c_{ij}^{π} are defined by

$$c_{ij}^{\pi} = d_{\pi} \int_{G} f(g) \overline{f_{ij}^{\pi}(g)} dg \tag{302}$$

Theorem 11 (Section 2.3.7, Theorem 1. [140]). *Any continuous function from a linear compact group G can be uniformly approximated by linear combinations of matrix coefficients f_{ij}^{π}, for $\pi \in \widehat{G}$, and $0 \geq i, j \leq d_{\pi}$, where d_{π} is the dimension of the representation space V_{π}.*

Theorem 12 (Section 2.3.9, Theorem 1. [140]). *Let H be a subgroup of G. Let $L_H^2(G)$ be the subspace of right H-invariant functions of $L^2(G)$. Then any functions $f \in L_H^2(G)$ can be expanded in the following way:*

$$f(g) = \sum_{\alpha \in \widehat{G/H}} \sum_{i=1}^{d_{\alpha}} \sum_{j=1}^{k_{\alpha}} c_{ij}^{\alpha} f_{ij}^{\alpha}(g) \tag{303}$$

The question is then what is $C^{\infty}(G/K)$, $L^2(G/K)$ and $C^0(G/K)$ in terms of the decompositions given above? $C(G/K)$ are all the right K-invariant functions, we see that the right action r when restricted to K is trivial. Meaning we have that:

Theorem 13 ([123]).

$$L^2(G/K) \cong \widehat{\bigoplus}_{\pi \in \widehat{G}} E_{\pi} \otimes (E_{\pi}^{*})^{K} \tag{304}$$

where E_{π}^{K} are representations of G such that $\pi(gk) = \pi(g)$ for all $k \in K$.

We see this by making use of the fact that $r_k f = f$ for all $k \in K$, and using the expansion

$$r_k f(g) = f(gk) = \sum_{j,m,m'} \langle j, m' | \pi^j(gk) | j, m \rangle = \sum_{j,m,m'} \langle j, m' | \pi^j(g) \pi^j(k) | j, m \rangle \tag{305}$$

$$= \sum_{j,m,n} \langle j, m' | \pi^j(g) | j, n \rangle \langle j, n | \pi^j(k) | j, m \rangle \tag{306}$$

, where repeated indices are summed over, j labels the irreps π and m, m' index an orthonormal basis of the representation. As we require this to be equal to f, we require that $\langle j, n | \pi^j(k) | j, m \rangle = \delta_{nm}$, which is equivalent to saying that $\pi^j(k) | j, m \rangle = 1$ for every m.

Example 9. *We often see statements about "the functions on the sphere can be decomposed in spherical harmonics". This type of statement is a restatement of the above construction, where $S^2 = SO(3)/SO(2) \cong SU(2)/U(1)$, and then the matrix coefficients can be related to the spherical harmonics, in the following way: we write $g \in SU(2)$ in the following parametrisation $g = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$ where J_i are the standard hermitian generators for the Lie algebra $\mathfrak{su}(2)$. For $S^2 \cong SU(2)/U(1)$, we take the representation of $U(1) = diag(e^{i\theta}, e^{-i\theta}) = exp(-i\theta J_z)$. The finite-dimensional irreducible unitary representations of $su(2)$ are indexed by a non-negative integer l and have dimension $2l + 1$. We can find a basis of each representation denoted by $\{|j, m\rangle\}_{m=-l}^{l}$ so we have that $\pi_j(J_z)|j, m\rangle = m|j, m\rangle$. In order for $\pi_j(diag(e^{i\theta}, e^{-i\theta})|j, m\rangle = \pi_j(e^{-i\theta J_z})|j, m\rangle = |j, m\rangle$, we need that $m = 0$.*

$$f^{\pi_j}_{m0}(g) = D^J_{m0}(\alpha, \beta, \gamma) = (-1)^m \sqrt{\frac{4\pi}{2j+1}} Y_l^{-m}(\alpha, \theta) \tag{307}$$

So if we let V_l be the vector space spanned by all the spherical harmonics Y_l^m, where $m = -l, -l+1, ..., l-1, l$. Then we can write that

$$C(G) = \bigoplus_{l=1}^{\infty} V_l. \tag{308}$$

BIBLIOGRAPHY

[1] Albert Einstein. The foundation of the general theory of relativity. *Annalen Phys.*, 49(7):769–822, 1916.

[2] B. P. Abbott et al. Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett.*, 116:061102, Feb 2016.

[3] Rudolf Haag and Daniel Kastler. An Algebraic Approach to Quantum Field Theory. *Journal of Mathematical Physics*, 5(7):848–861, 1964.

[4] Rudolf Haag. *Local quantum physics: fields, particles, algebras*. Springer, Berlin, 2nd rev. and enl. ed edition, 1996.

[5] Matthew D Schwartz. *Quantum field theory and the standard model*. Cambridge University Press, 2014.

[6] Michael B. Green, John H. Schwarz, and Edward Witten. *Superstring Theory: 25th Anniversary Edition*, volume 1 and 2 of *Cambridge Monographs on Mathematical Physics*. Cambridge University Press, 2012.

[7] Frederik Denef and Michael R Douglas. Distributions of flux vacua. *Journal of High Energy Physics*, 2004(05):072–072, 2004.

[8] Washington Taylor and Yi-Nan Wang. The F-theory geometry with most flux vacua. *Journal of High Energy Physics*, 2015(12):1–21, 2015.

[9] Michael R. Douglas. The String Theory Landscape. *Universe*, 5(7):176, 2019.

[10] Carlo Rovelli and Francesca Vidotto. *Covariant Loop Quantum Gravity*. Cambridge University Press, 2009.

[11] J Ambjorn, J Jurkiewicz, and R Loll. The universe from scratch. *Contemporary Physics*, 47(2):103–117, 2006.

[12] J Ambjorn, J Jurkiewicz, and R Loll. Reconstructing the Universe. *Physical Review D*, 72(6):064014, 2005.

[13] R Loll. Quantum gravity from causal dynamical triangulations: a review. *Classical and Quantum Gravity*, 37(1):013002, 2019.

[14] Luca Bombelli, Joohan Lee, David Meyer, and Rafael D Sorkin. Space-time as a causal set. *Physical Review Letters*, 59(5):521–524, 1987.

[15] Sumati Surya. The causal set approach to quantum gravity. *Living Reviews in Relativity*, 22(1):5, 2019.

[16] T Regge. General relativity without coordinates. *Il Nuovo Cimento*, 19(3):558–571, 1961.

[17] Jürgen Riedler, Wolfgang Beirl, Elmar Bittner, Alf Hauke, Peter Homolka, and Harald Markum. Phase structure and graviton propagators in lattice formulations of four-dimensional quantum gravity. *Classical and Quantum Gravity*, 16(4):1163–1173, 1999.

[18] Herbert W Hamber. Phases of simplical quantum gravity in four dimensions estimates for the critical exponents. *Nuclear Physics B*, 400(1-3):347–389, 1993.

[19] John W Barrett and Louis Crane. Relativistic spin networks and quantum gravity. *Journal of Mathematical Physics*, 39(6):3296–3302, 1998.

[20] G. Ponzano and T. Regge. Semiclassical limit of racah coefficients. *pp 1-58 of Spectroscopic and Group Theoretical Methods in Physics. Block, F. (ed.). New York, John Wiley and Sons, Inc., 1968.*, 10 1969.

[21] Justin Roberts. Classical 6j-symbols and the tetrahedron. *Geometry and Topology*, 3:21–66, 1999.

[22] Carlo Rovelli. Black Hole Entropy from Loop Quantum Gravity. *Physical Review Letters*, 77(16):3288–3291, 1996.

[23] Ernesto Frodden, Amit Ghosh, and Alejandro Perez. Quasilocal first law for black hole thermodynamics. *Physical Review D*, 87(12):121503, 2013.

[24] A Ashtekar, J Baez, A Corichi, and K Krasnov. Quantum Geometry and Black Hole Entropy. *Physical Review Letters*, 80(5):904–907, 1998.

[25] Dionigi M T Benincasa and Fay Dowker. Scalar Curvature of a Causal Set. *Physical Review Letters*, 104(18):181301, 2010.

[26] William J. Cunningham and Sumati Surya. Dimensionally restricted causal set quantum gravity: Examples in two and three dimensions, 2019.

[27] Lisa Glaser, Denjoe O'Connor, and Sumati Surya. Finite size scaling in 2d causal set quantum gravity. *Classical and Quantum Gravity*, 35(4):045006, 2018.

[28] Sumati Surya. Evidence for the continuum in 2D causal set quantum gravity. *Classical and Quantum Gravity*, 29(13):132001, 2012.

[29] W. Heisenberg. Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen. *Zeitschrift für Physik*, 33(1):879–893, 1925.

[30] M. Born and P. Jordan. Zur Quantenmechanik. *Zeitschrift für Physik*, 34(1):858–888, 1925.

[31] M Born, W Heisenberg, and P Jordan. Zur Quantenmechanik. II. *Zeitschrift for Physik*, 35(8-9):557–615, 1926.

[32] I. Gelfand and M. Neumark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):197–213, 1943.

[33] Jean-Pierre Serre. Faisceaux algébriques cohérents. *Ann. of Math. (2)*, 61:197–278, 1955.

[34] Richard G. Swan. Vector bundles and projective modules. *Trans. Amer. Math. Soc.*, 105:264–277, 1962.

[35] Alain Connes. On the spectral characterization of manifolds. *J. Noncommut. Geom.*, 7(1):1–82, 2013.

[36] Attilio Palatini. Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton. *Rendiconti del Circolo Matematico di Palermo*, 43(1):203–212, 1919.

[37] Jerzy F Plebański. On the separation of Einsteinian substructures. *Journal of Mathematical Physics*, 18(12):2511–2520, 1977.

[38] Sören Holst. Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action. *Physical Review D*, 53(10):5966–5969, 1996.

[39] Ali H. Chamseddine and Alain Connes. The spectral action principle. *Comm. Math. Phys.*, 186(3):731–750, 1997.

[40] Matilde Marcolli. *Noncommutative Cosmology*. WORLD SCIENTIFIC, 2018.

[41] Ali H. Chamseddine, Alain Connes, and Matilde Marcolli. Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.*, 11(6):991–1089, 2007.

[42] John W. Barrett. Lorentzian version of the noncommutative geometry of the Standard Model of particle physics. *J. Math. Phys.*, 48(1):012303, 7, 2007.

[43] Ali H. Chamseddine and Alain Connes. Resilience of the spectral standard model. *J. High Energy Phys.*, 2012(9), 2012.

[44] Christopher Estrada and Matilde Marcolli. Asymptotic safety, hypergeometric functions, and the Higgs mass in spectral action models. *Int. J. Geom. Methods Mod. Phys.*, 10(7):1350036, 30, 2013.

[45] Francesco D'Andrea, Fedele Lizzi, and Joseph C Varilly. Metric Properties of the Fuzzy Sphere. *Letters in Mathematical Physics*, 103(2):183–205, oct 2012.

[46] John W Barrett. Matrix geometries and fuzzy spaces as finite spectral triples. *Journal of Mathematical Physics*, 56(8):82301, aug 2015.

[47] James A. B. Gaunt. *Aspects of the fuzzy torus*. PhD thesis, The University of Nottingham, 11 2018.

[48] John W. Barrett and James Gaunt. Finite spectral triples for the fuzzy torus. *arXiv e-prints*, page arXiv:1908.06796, Aug 2019.

[49] John Madore. The fuzzy sphere. *Classical and Quantum Gravity*, 9(1):69–87, jan 1992.

[50] Alain Connes. Noncommutative geometry and reality. *J. Math. Phys.*, 36(11):6194–6231, 1995.

[51] H. Grosse and P. Prešnajder. The Dirac operator on the fuzzy sphere. *Lett. Math. Phys.*, 33(2):171–181, 1995.

[52] Ursula Carow-Watamura and Satoshi Watamura. Chirality and Dirac operator on noncommutative sphere. *Communications in Mathematical Physics*, 183(2):365–382, 1997.

[53] Ursula Carow-Watamura and Satoshi Watamura. Differential Calculus on Fuzzy Sphere and Scalar Field. *International Journal of Modern Physics A*, 13(19):3235–3243, 1998.

[54] Balachandran, A P and Govindarajan, T R and Ydri, B. The Fermion Doubling Problem and Noncommutative Geometry. *Modern Physics Letters A*, 15(19):1279–1286, jun 2000.

[55] A. Pinzul and A. Stern. Dirac operator on the quantum sphere. *Physics Letters B*, 512(1-2):217–224, jul 2001.

[56] Badis Ydri. Noncommutative chiral anomaly and the Dirac-Ginsparg-Wilson operator. *Journal of High Energy Physics*, 2003(08):046–046, 2003.

[57] Ursula Carow-Watamura and Satoshi Watamura. Noncommutative Geometry and Gauge Theory on Fuzzy Sphere. *Communications in Mathematical Physics*, 212(2):395–413, 2000.

[58] Ursula Carow-Watamura, Harold Steinacker, and Satoshi Watamura. Monopole bundles over fuzzy complex projective spaces. *Journal of Geometry and Physics*, 54(4):373–399, 2005.

[59] AP Balachandran, Brian P Dolan, J Lee, X Martin, and Denjoe OConnor. Fuzzy complex projective spaces and their star-products. *Journal of Geometry and Physics*, 43(2-3):184–204, 2002.

[60] A.P Balachandran and Pramod Padmanabhan. Spin j Dirac Operators on the Fuzzy 2-Sphere. *Journal of High Energy Physics*, 2009(09):120–120, sep 2009.

[61] John W. Barrett and Lisa Glaser. Monte Carlo simulations of random non-commutative geometries. *J. Phys. A*, 49(24):245001, 27, 2016.

[62] H B Lawson and Marie-Louise Michelsohn. *Spin Geometry*, volume 38. Princeton University Press, Princeton, NJ, jan 1989.

[63] Lisa Glaser. Scaling behaviour in random non-commutative geometries. *J. Phys. A*, 50(27):275201, 18, 2017.

[64] John W. Barrett, Paul Druce, and Lisa Glaser. Spectral estimators for finite non-commutative geometries. *J. Phys. A*, 52(27):275203, 33, 2019.

[65] Alain Connes. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.

[66] Abel B. Stern. Finite-rank approximations of spectral zeta residues. *Lett. Math. Phys.*, 109(3):565–577, 2019.

[67] Gunther Cornelissen and Aristides Kontogeorgis. Distances in spaces of physical models: partition functions versus spectra. *Letters in Mathematical Physics*, 107(1):129–144, January 2017.

[68] P A M Dirac. The Quantum Theory of the Electron. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 117(778):610–624, 1928.

[69] Thomas Friedrich. Dirac Operators in Riemannian Geometry. *Graduate Studies in Mathematics*, 25(25):195, aug 2000.

[70] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.

[71] Alain Connes. Gravity coupled with matter and the foundation of non-commutative geometry. *Comm. Math. Phys.*, 182(1):155–176, 1996.

[72] Latham Boyle and Shane Farnsworth. A new algebraic structure in the standard model of particle physics. *Journal of High Energy Physics*, 2018:1–32, 2018.

[73] Walter D van Suijlekom. *Noncommutative Geometry and Particle Physics*. Springer, jul 2014.

[74] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002.

[75] Branimir Ćaćić. Moduli Spaces of Dirac Operators for Finite Spectral Triples. In *Quantum Groups and Noncommutative Spaces*, pages 9–68. Vieweg+Teubner, 2011.

[76] Serge Lang. *Graduate Texts in Mathematics: Algebra*. Springer, 2002.

[77] Benjamin Steinberg. *Representation Theory of Finite Groups*. Springer New York, 2012.

[78] Nicolas Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation [MR0979982 (90d:00002)].

[79] Harald Grosse, Christian W Rupp, and Alexander Strohmaier. Fuzzy line bundles, the Chern character and topological charges over the fuzzy sphere. *Journal of Geometry and Physics*, 42(1-2):54–63, 2002.

[80] E Harikumar, Amilcar R Queiroz, and Paulo Teotonio-Sobrinho. Dirac operator on the q-deformed fuzzy sphere and its spectrum. *Journal of High Energy Physics*, 2006(09):037–037, 2006.

[81] Gaetano Fiore and Francesco Pisacane. Fuzzy circle and new fuzzy sphere through confining potentials and energy cutoffs. *Journal of Geometry and Physics*, 132(Phys. Rev. 71 1947):423–451, 2018.

[82] Brian P Dolan, Idrish Huet, Seán Murray, and Denjoe O'Connor. Noncommutative vector bundles over fuzzy CPN and their covariant derivatives. *Journal of High Energy Physics*, 2007(07):007–007, 2007.

[83] Brian P Dolan, Idrish Huet, Seán Murray, and Denjoe O'Connor. A universal Dirac operator and noncommutative spin bundles over fuzzy complex projective spaces. *Journal of High Energy Physics*, 2008(03):029–029, 2008.

[84] A Chaney and A Stern. Fuzzy C P 2 spacetimes. *Physical Review D*, 95(4):046001, 2017.

[85] Joakim Arnlind, Martin Bordemann, Laurent Hofer, Jens Hoppe, and Hidehiko Shimada. Fuzzy Riemann surfaces. *Journal of High Energy Physics*, 2009(06):047–047, 2009.

[86] Marc Rieffel. C -algebras associated with irrational rotations. *Pacific Journal of Mathematics*, 93(2):415–429, 1981.

[87] Tomasz Brzeziński. On the Smoothness of the Noncommutative Pillow and Quantum Teardrops. *Symmetry, Integrability and Geometry: Methods and Applications*, 2013.

[88] Réamonn Ó Buachalla. Quantum Bundle Description of the Quantum Projective Spaces. *Communications in Mathematical Physics*, 316(2):345–373, 2011.

[89] Réamonn Ó Buachalla. Noncommutative complex structures on quantum homogeneous spaces. *Journal of Geometry and Physics*, 99:154–173, 2016.

[90] Réamonn Ó Buachalla. Noncommutative Kähler structures on quantum homogeneous spaces. *Advances in Mathematics*, 322:892–939, 2017.

[91] John Madore. *An introduction to noncommutative differential geometry and its physical applications*, volume 257 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, second edition, 1999.

[92] Marc A. Rieffel. *Gromov-Hausdorff distance for quantum metric spaces. Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance*. American Mathematical Society, Providence, RI, 2004.

[93] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge University Press, London-New York, 1973. Cambridge Monographs on Mathematical Physics, No. 1.

[94] A. Trautman. The Dirac operator on hypersurfaces. *Acta Phys. Polon. B*, 26(7):1283–1310, 1995.

[95] Frédéric Latrémolière. The quantum Gromov-Hausdorff propinquity. *Trans. Amer. Math. Soc.*, 368(1):365–411, 2016.

[96] Jayewardena. Schwinger model on Sˆ2. *Helvetica Physica Acta*, 1988.

[97] Tomasz Brzezinski, Nicola Ciccoli, Ludwik Dabrowski, and Andrzej Sitarz. Twisted reality condition for Dirac operators. *Mathematical Physics, Analysis and Geometry*, 19(3):16, 2016.

[98] Tomasz Brzezinski, Ludwik Dabrowski, and Andrzej Sitarz. On twisted reality conditions. *Letters in Mathematical Physics*, 109(3):643–659, 2018.

[99] Christian Bär. Dependence of the Dirac spectrum on the Spin structure. *Global analysis and harmonic analysis (Marseille-Luminy, 1999)*, 4:17–33, 2000.

[100] H B Nielsen and M Ninomiya. A no-go theorem for regularizing chiral fermions. *Physics Letters B*, 105(2-3):219–223, 1981.

[101] H.B. Nielsen and M. Ninomiya. Absence of neutrinos on a lattice (I). Proof by homotopy theory. *Nuclear Physics B*, 185(1):20–40, 1981.

[102] Madan Lal Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.

[103] M. E. J. Newman and G. T. Barkema. *Monte Carlo methods in statistical physics*. The Clarendon Press, Oxford University Press, New York, 1999.

[104] B Mandelbrot. How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension. *Science*, 156(3775):636–638, 1967.

[105] A. Connes and H. Moscovici. The local index formula in noncommutative geometry. *Geom. Funct. Anal.*, 5(2):174–243, 1995.

[106] Yuri I. Manin. The notion of dimension in geometry and algebra. *Bull. Amer. Math. Soc. (N.S.)*, 43(2):139–161, 2006.

[107] Peter B Gilkey. The Spectral geometry of a Riemannian manifold. *J. Diff. Geom.*, 10(4):601–618, jan 1975.

[108] John Roe. *Elliptic operators, topology and asymptotic methods*, volume 395 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, second edition, 1998.

[109] J Milnor. Eigenvalues of the Laplace operator on certain manifolds. *Proceedings of the National Academy of Sciences of the United States of America*, 51:542, jan 1964.

[110] Mark Kac. Can one hear the shape of a drum? *Amer. Math. Monthly*, 73(4, part II):1–23, 1966.

[111] Christian Bar. The Dirac operator on space forms of positive curvature. *Journal of the Mathematical Society of Japan*, 48(1):69–83, 1996.

[112] Alain Connes. A unitary invariant in Riemannian geometry. *Int. J. Geom. Methods Mod. Phys.*, 5(8):1215–1242, 2008.

[113] José M. Gracia-Bondía, Joseph C. Várilly, and Héctor Figueroa. *Elements of noncommutative geometry*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, 2001.

[114] Peter Greiner. An asymptotic expansion for the heat equation. *Archive for Rational Mechanics and Analysis*, 41:163–218, 1971.

[115] R T Seeley. *Complex Powers of an Elliptic Operator*, volume 10. American Mathematical Society, Providence, Rhode Island, jan 1967.

[116] Peter B Gilkey. Invariance theory, the heat equation and the Atiyah-Singer index theorem. *Studies in Advanced Mathematics*, pages x–516, jan 1995.

[117] D. V. Vassilevich. Heat kernel expansion: user's manual. *Phys. Rep.*, 388(5-6):279–360, 2003.

[118] Audrey Terras. *Harmonic analysis on symmetric spaces and applications. I*. Springer-Verlag, New York, 1985.

[119] J. Ambjørn, J. Jurkiewicz, and R. Loll. The spectral dimension of the universe is scale dependent. *Phys. Rev. Lett.*, 95:171301, Oct 2005.

[120] S Carlip. Dimension and dimensional reduction in quantum gravity. *Classical and Quantum Gravity*, 34(19):193001, sep 2017.

[121] Gianluca Calcagni, Daniele Oriti, and Johannes Thürigen. Spectral dimension of quantum geometries. *Classical Quantum Gravity*, 31(13):135014, 33, 2014.

[122] Brian Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2015. An elementary introduction.

[123] Mark R. Sepanski. *Compact Lie groups*, volume 235 of *Graduate Texts in Mathematics*. Springer, New York, 2007.

[124] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Academic Press, 1978.

[125] Andreas Arvanitoyeorgos. *An introduction to Lie groups and the geometry of homogeneous spaces*, volume 22 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2003. Translated from the 1999 Greek original and revised by the author.

[126] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. II*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996.

[127] Andreas Čap and Jan Slovák. *Parabolic geometries. I*, volume 154 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2009. Background and general theory.

[128] Nolan R. Wallach. *Harmonic analysis on homogeneous spaces*. Marcel Dekker, Inc., New York, 1973. Pure and Applied Mathematics, No. 19.

[129] Michel Cahen, Simone Gutt, and Andrzej Trautman. Spin structures on real projective quadrics. *J. Geom. Phys.*, 10(2):127–154, 1993.

[130] Christian Bär. The Dirac operator on homogeneous spaces and its spectrum on 3-dimensional lens spaces. *Arch. Math. (Basel)*, 59(1):65–79, 1992.

[131] Marc A. Rieffel. Dirac operators for coadjoint orbits of compact Lie groups. *Münster J. Math.*, 2:265–297, 2009.

[132] Marc A. Rieffel. A global view of equivariant vector bundles and Dirac operators on some compact homogeneous spaces, 2008.

[133] Jens Kaad and Roger Senior. A twisted spectral triple for quantum SU(2). *Journal of Geometry and Physics*, 62(4):731–739, 2011.

[134] Pierre Martinetti. Twisted spectral geometry for the standard model. *Journal of Physics: Conference Series*, 626(1):012044, 2015.

[135] Magnus Goffeng, Bram Mesland, and Adam Rennie. Untwisting twisted spectral triples. *International Journal of Mathematics*, 30(14):1950076, 2019.

[136] Ludwik Dabrowski and Francesco D'Andrea and Adam M. Magee. Twisted reality, second order condition and Hodge-de Rham spectral triples, 2019.

[137] Ludwik Dabrowski and Andrzej Sitarz. Multiwisted real spectral triples, 2019.

[138] A Devastato, S Farnsworth, F Lizzi, and P Martinetti. Lorentz signature and twisted spectral triples. *Journal of High Energy Physics*, 2018(3):89, 2018.

[139] Pierre Martinetti and Devashish Singh. Lorentzian fermionic action by twisting euclidean spectral triples, 2019.

[140] N. Ja. Vilenkin and A. U. Klimyk. *Representation of Lie Groups and Special Functions*. Springer Netherlands, 1991.