Second Year Report — Noncommutative Geometry and Quantum Gravity

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1 Literature Review

The aim of this literature review is to outline the setting in which my current work lies. To do this, a brief physical motivation is provided to explain why the study of noncommutative geometries is interesting to those who wish to achieve a theory of quantum gravity. Then a more detailed review of finite noncommutative geometry as noncommutative analogues to well known spaces will be presented.

One of the problems to overcome with a quantum theory of gravity is that the current framework for gravity requires the space we, and all of physics, exist in to be described by a manifold. Manifolds by construction allow for infinitesimally small regions of space to be a well defined construct. And all of the physics we do over this manifold, is required to deal with these infinitesimally small regions. It is this feature that causes the incompatibility between our understanding of gravity and out quantum theory of the other forces we know of. Some of the more well known approaches to quantum gravity aim to fix this problem by make space discretised. By making the space on which everything exists on to be built up from some smallest building blocks, they circumvent this problem. However, different problems arise and there is much research into producing a framework built on these principles that is satisfactory [1]. The other main approach is string theory, which was found to be connected to gravity by the fact that one of the string modes corresponds to the graviton, the hypothetical force boson for gravity. However there are issues with string theory, which include its reliance on supersymmetry. A proposed new symmetry that means every particles currently known has a supersymmetric partner. However, no evidence has of supersymmetry has been seen and this coupled with the lack of predictive power of string theory make it a undesirable framework to currently research.

Noncommutative geometry is being investigated as a model for quantum gravity for two main reasons. The first is that the standard model of particle physics can be expressed as a finite noncommutative geometry [2]. Which then under the framework of spectral triples, puts general relativity and particle physics in the same framework (see below for how manifolds are expressed as noncommutative geometries) making unification of the forces a realistic future of the theory. Also the idea of noncommutative geometry is to make the algebra of functions to be noncommutative, which would make the positions on the space no longer commute, making the notion of a point ill-defined. This is an exciting feature for a potential theory of gravity, because it would make our understanding of quantum mechanics and gravity both regularise all of the divergences that usually appear in the standard way of trying to make gravity a quantum field theory.

A problem with discretised models such as loop quantum gravity and causal set theory is their lack of familiar structure, and a lot of work goes into setting up various structures that exist in normal differential geometric approaches. This problem is also present in the framework of noncommutative geometry. A brief outline of the fundamentals of noncommutative geometry is given below in Appendix A.

Noncommutative geometry is an extension of normal *commutative geometry* where the commutativity refers to the algebra of continuous functions defined on a topological space always being a commutative algebra. For instance, a Riemannian manifold always has a commutative algebra associated to it, $C^\infty(M)$, which is all of the smooth functions defined everywhere on the manifold, where the algebra structure is provided by *pointwise* multiplication and addition. I.e given two functions $f, g\colon M \to \mathbb{R}$ we define fg and $f+g$ as follows

$$(fg)(x) = f(x)g(x) \tag{1}$$
$$(f+g)(x) = f(x) + g(x). \tag{2}$$

The task of noncommutative geometry is to try and explore spaces where this algebra of smooth functions is no longer commutative.

Originally the space was taken to be a topological spaces and then using the Gelfand duality between topological spaces and commutative C^*-algebras as a guideline, efforts to extend this duality to noncommutative algebras and seeing what sort of geometric objects resulted from the procedure was undertaken. It was also shown that there exists a duality between a compact Riemannian spin manifolds and an algebraic structure called a *commutative real spectral triple* [3]. An aim of noncommutative geometry is to try extend this duality to where the algebra (of smooth functions) involved in the spectral triple is noncommutative [4, 5]. Efforts to relax some of the other conditions on the manifold are being investigated, such as the work on Lorentzian spectral triples [6–11] and noncompact geoemtries [12, 13].

This is where the field branches into different subtopics, and where we will follow only one. Depending on what correspondence between the geometrical world and the algebraic world you want to focus on specifies which area of noncommutative geometry you end up work in. The focus for the rest of this report will be on the differential nature of geometries, which follows the work of Connes mentioned above[1].

1.0.1 Noncommutative analogues to manifolds

A lot of work has gone into trying to find noncommutative spectral triples that have the behaviour of a manifold[2]. The most famous example is that of the Fuzzy Sphere [4] which constructions a noncommutative spectral triple, where the algebra is a finite matrix algebra and the Dirac operator is invariant under the action of $\mathfrak{su}(2)$, the Lie algebra of $SO(3)$. The construction of the fuzzy sphere as a spectral triple is outlined in Appendix B.

The fuzzy sphere highlights one of the reasons why there is current research into noncommutative geometry as a quantum gravity pathway. There is an interpretation of the fuzzy sphere as being a sphere but with cutoff in the spectral harmonics which is described extensively in [14]. Given the algebra of functions $C(\mathbb{S}^2)$, we can decompose this space into a direct sum of irreducible

[1] See AppendixA for a the basics of spectral triples

[2] This is purposefully vague as you can't recover all the behaviours of a manifold and still be dealing with noncommutative geometry, so there is a choice in which behaviours you want to exhibit

representations of $\mathfrak{su}(2)$:

$$C(\mathbb{S}^2) \simeq \bigoplus_{l=0}^{\infty} V_l \tag{3}$$

where V_l is the vector space underlying the irreducible representation of $su(2)$ with the highest weight $l \in \mathbb{N}$ which is spanned by the spherical harmonics $Y_{l,m}$. We can then impose a cutoff in the energy spectrum by ignore all but the first $n+1$ representations in the decomposition of $C(\mathbb{S}^2)$. Thus, in the fuzzy sphere's spectral triple we can take the *fuzzy spherical harmonics*[3] $\hat{Y}_{l,m}$ to be the generators of $M_{n+1}(\mathbb{C})$, where[4] $l < n$. We can then decompose the algebra into

$$\mathcal{A}_n \simeq \bigoplus_{l=0}^{n} V_l \tag{4}$$

As the spherical harmonics correspond to higher angular momentum modes, the fact that there is a *cutoff in* l can be interpreted as having a maximum angular momentum and thus energy for the space. As we can view a maximum energy as an equivalent minimum region we can probe, the fuzzy sphere can be viewed as having a minimal renderable distance, i.e. a Planck length. The implications of a planck length being a natural outcome of requiring the underlying space to be noncommutative is a very appealing property and the idea is that a noncommutative analogue to a spacetime will provide a good model for quantum gravity.

2 Overview of attained results

One of the main drawbacks of noncommutative geometry is the lack of any familiar differential calculus. With the notion of coordinates being ill defined, derivatives and integrals are also hard to express. However, there is a mechanism for integration over the noncommutative geometry and also there are frameworks for a differential calculus, but they have some caveats and intricacies to work out (see [16] for a review of the developments.). However, the other tools we usually have in geometry to describe certain spaces are few and far between. The notions of curvature and dimension are all lost in the translation. So much of the research conducted is to create spaces that have some familiar traits, and also to develop measurements we can take of these spaces that give us a way to categorise them. So far, the spaces well described all have some Lie algebra symmetry underlying them. The easiest to grasp is that of a fuzzy sphere mentioned before. Determining what dimension the fuzzy sphere is a non-trivial question due to the fact that its algebra is finite dimensional, many of the proposed dimensional measures proposed do not yield results for such spaces. So methods of determining the dimension of a fuzzy sphere has been explored and also applied to the random fuzzy geometries of [17] and also a fuzzy torus being developed by [18]. These methods are based on the spectral properties of the heat kernel trace and the spectral zeta function.

3 Research Plan for the next 12 months

1. Finish Spectral Dimension and Zeta function paper with John and Lisa August/September 2017

[3] These are matrix versions of the spherical harmonics, there precise construction can be found in [15].

[4] Note that the reason $l < n$ not $l < n+1$ is because the index l starts at zero. So there are still $n+1$ generators for $M_{n+1}(\mathbb{C})$

2. Research Dirac operators on coadjoint orbits (and their generalisation when they don't exist in the normal sense)

3. Research the difference between Lie derivative and covariant derivative on spinor fields. This will be closely related to the coadjoint orbit work.

4 Time table for drafting Thesis

Once the paper for the zeta function is submitted, I'll start writing up the section of zeta functions and all the necessary details. If the submission is delayed, I will start writing the thesis section in September/October 2017.

Once I have a section on the zeta function, this will lead me to introduce various topics. Which I will use to flesh out the document. So as December arrives, I'll be writing up the basics needed, trying to not be too mathematically heavy where I can, otherwise it will be a long arduous document. By April 2018 I should have produced some new work on the topics mentioned above, which again will lead me to writing up new background and new chapters. Which will start around April if all goes well. The finishing of the thesis will be set into full motion in August/September 2018, when all research will stop and I know what topics I will be dealing with.

5 A substantial piece of work demonstrating clear and coherent maths.

5.1 Spectral Zeta Functions, Heat Kernels and Geometry

The aim of spectral geometry as a whole is to try and recover the geometric information of various types of space (Riemannian manifolds, metric spaces etc) by examining the spectrum of their various operators. The most famous example is by studying the Laplacian of a Riemannian manifold and trying to determine the dimension, volume, integral of the scalar curvature etc. For operators on finite dimensional vector spaces or for compact operators the spectrum is just the set of eigenvalues of the operator. However for more complicated objects, like Riemannian manifolds or when the operator acts on an infinite-dimensional vector space, the spectrum contains the eigenvalues and other values which are not eigenvalues.

The operator we are concerned with is that of the Dirac Operator which can be defined local on any spin manifold as

$$\not{D} = ie^{\mu}_{a}\gamma_{a}\nabla_{\mu} \tag{5}$$

where e^{μ}_{a} are vielbeins[5] and γ^{a} are the flat gamma matrices which generalise the Pauli matrices and Dirac matrices to higher dimensions. The main motivation for studying the Dirac operator over other operators is because of the Reconstruction theorem by Alain Connes. This states that given a commutative real spectral triple you can recover a unique compact manifold and Riemannian metric such that the algebra is the algebra of functions, the Hilbert space is the twice integrable spin sections and the self adjoint operator is the Dirac operator. This is a powerful spectral theorem, as such statements for the laplace operator lack uniqueness due to isospectral laplacians [19]. From here onwards we will refer to a smooth compact Riemannian manifold as just

[5] These are sometimes refered to as frame fields in the context of general relativity and in general they are a local orthonormal basis for the tangent bundle.

a manifold. A method to extract information about a manifold from its Dirac operator spectrum is by studying the spectral zeta functions. Traditionally the Laplace operator is the object studied via spectral zeta functions, however much of the theory is valid for operators of Laplace-type. These are second order differential operators,P, such that $P = \Delta + E$, where E is a zeroth order differential operator. If we have a first order operator D such that D^2 is of Laplace type then we say that D is of Dirac-type. We will be examining the spectral zeta function (SPζ) of the Dirac operator squared, which can be see to be of Laplace-type from the Lichnerowicz formula [20, 21]. Let $\{\lambda_i\}$ be the set of non zero eigenvalues of a Dirac operator, D. Then the spectral zeta function of the Dirac operator is defined as[6]:

$$\zeta_{D^2}(s) = \mathrm{Tr}\left(\left(D^2\right)^{-s}\right) = \sum_i \left(\lambda_i^2\right)^{-s} . \tag{6}$$

To extract geometry from the SPζ we need to draw on another spectral invariant called the Heat Kernel trace[7] which is defined as follows

$$K(t) = \mathrm{Tr}(\exp\left(-tD^2\right)). \tag{7}$$

We can then make use of the fact that the spectral zeta function is the inverse Mellin transform of the spectral heat kernel:

$$\zeta_{D^2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t) \mathrm{d}t \tag{8}$$

This expression can be inverted to express the heat kernel in terms of the zeta function:

$$K(t) = \frac{1}{2\pi i} \oint ds\ t^{-s} \Gamma(s) \zeta_{D^2}(s) \tag{9}$$

where the contour surrounds all the poles of the integrand. Note that these transforms work in the case of a finite spectral triple and then the poles in (9) are just those of the gamma function. The Heat Kernel for a Laplace type operator has an asymptoptic expansion as follows [22–24]:

$$K(t) \simeq t^{-d/2}(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots) \tag{10}$$

where d is the dimension of the underlying space. Using this expansion, we can relate the residues of the integrand of (9) to the expansion coefficients a_k. We find that

$$a_k = Res_{s=\frac{d-k}{2}}(\Gamma(s)\zeta_{D^2}(s)) \tag{11}$$

where $\Gamma(s)$ is Euler's gamma function. It is known that the expansion coefficients are only non zero for even values of k [25] and for even k they can be expressed in terms of local geometrical invariants [26, 27]. The first two coefficients take the following geometric form:

$$a_0(D^2) = \frac{1}{(4\pi)^{\frac{d}{2}}} \mathrm{Tr}(Id) \int_M \mathrm{d}^d x \sqrt{g} \tag{12}$$

$$a_2(D^2) = -\frac{1}{(4\pi)^{\frac{d}{2}}} \frac{1}{12} \mathrm{Tr}(Id) \int_M \mathrm{d}^d x \sqrt{g} R \tag{13}$$

[6]The conditions of the space in order for the zeta function to be expressed as a sum are that it's compact in terms of manifolds. However, for the finite spectral triples we aim to explore, the sum is also well defined.

[7]Often shortened to just Heat Kernel

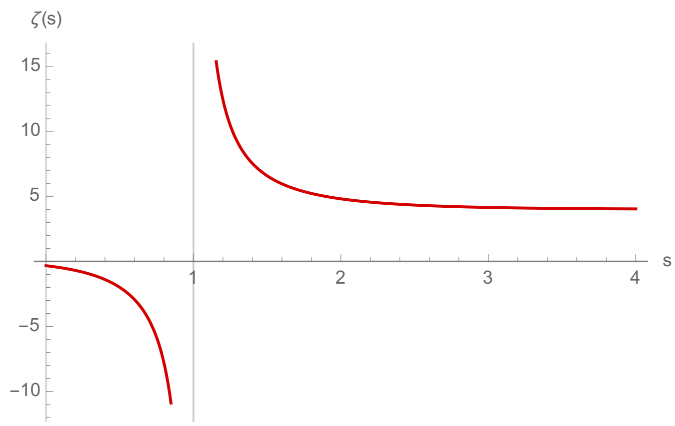


Figure 1: The SPζ for the continuum 2-sphere. The pole at $s = d/2 = 1$ indicates the dimension.

where here R is the Ricci scalar and the trace is over the spinor space. So we using equations (11)-(13) we can use the residues of the zeta function to determine various quantities such as the volume and curvature. Explicitly

$$Vol(M) = \frac{(4\pi)^{d/2}}{\mathrm{Tr}(Id)} Res_{s=\frac{d}{2}}(\Gamma(s)\zeta_{D^2}(s)) \tag{14}$$

$$\int_M \mathrm{d}^d x\sqrt{g}R = -12\frac{(4\pi)^{d/2}}{\mathrm{Tr}(Id)} Res_{s=\frac{d}{2}-1}(\Gamma(s)\zeta_{D^2}(s)) \tag{15}$$

For 2d manifolds, like the sphere which we use as a benchmark, the spinors are two dimensional, which gives[8] $Vol(M) = 2\pi Res_{s=1}(\zeta_{D^2}(s))$. Also for 2d manifolds the Ricci scalar is twice the Gaussian curvature, and we can make use of the Gauss-Bonnet theorem to related the second heat kernel coefficient to the Euler characteristic of the space.

$$\chi(M) = -6Res_{s=0}(\Gamma(s)\zeta_{D^2}(s)) = -6\zeta_{D^2}(0) \tag{16}$$

We will use these results from the spectral geometry of manifolds as a guideline for the investigations into the spectral geometry of finite noncommutative geometries. For a manifold, the spectrum is infinite, and the resulting zeta function (specifically its analytic continuation) has poles at $s = \frac{d}{2} - \nu$ for $\nu \in \{0, 1, 2, \ldots, \lfloor d/2 \rfloor\}$.

An example of this is the continuum 2-sphere, for which the SPζ is proportional to the Riemann ζ function with argument $2s-1$, and thus has a pole at $s = 1$ (c.f. Figure 1). For a fuzzy space, such as those outlined in [28], we have a finite number of eigenvalues, and hence the sum in (6) is finite, which automatically regularises the poles at $s = \frac{d}{2}, \cdots$. For a noncommutative analogue of a manifold, instead of a pole as $s = \frac{d}{2}$ we expect to see the series grow logarithmically. This is due to the fact that on a d dimensional manifold the n-th eigenvalue of the Dirac operator scales like $\lambda_n \sim n^{1/d}$ for large[9] n, so at $s = \frac{d}{2}$, the zeta function is the sum of n^{-1}, which diverges logarithmically. We expect fuzzy spaces to express this behaviour as we increase the size of the matrix algebra. If we have $N \gg 1$ eigenvalues, then the series diverges $\log(N)$. Using the fact that the zeta function series value at s = 0 precisely counts the number of eigenvalues when dealing with finite spectra. We can scale the zeta function by $\log(\zeta_{D^2}(0))^1$ and locate the value of s at

[8] The Gamma function is well defined at s=1, and thus all the residue lies in the zeta function. Also the value of the Gamma function at s=1 is 1.

[9] This comes from Weyl's law for the Dirac operator.

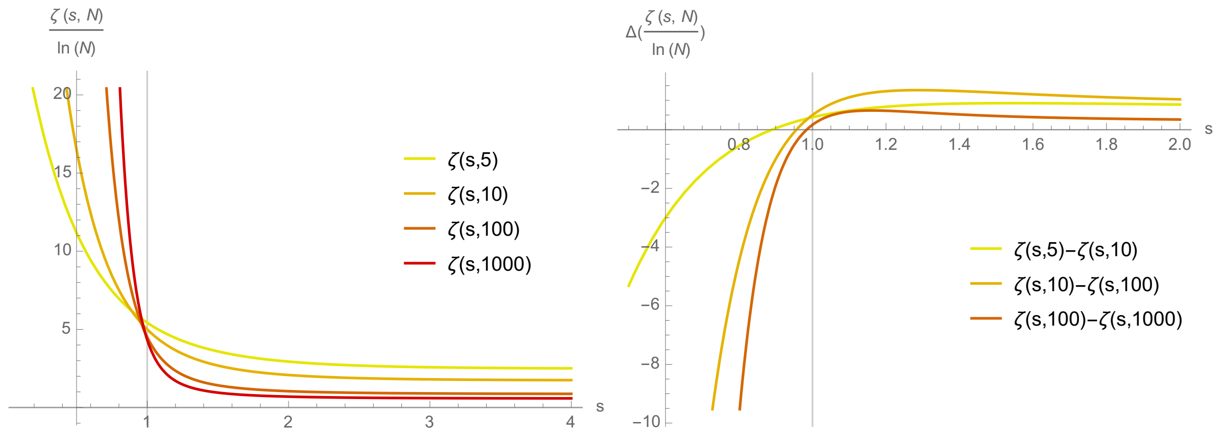


Figure 2: The MPζ for the fuzzy 2-sphere for different n.

which the zeta functions for various matrix sizes are equal, which would indicate the location of a pole in the infinite N limit. I.e the hypothesis is that the value of s where we have

$$\frac{\zeta_{D_n^2}(s)}{\log\left(\zeta_{D_n^2}(0)\right)} = \frac{\zeta_{D_m^2}(s)}{\log\left(\zeta_{D_m^2}(0)\right)} \tag{17}$$

will limit to the location of the pole of the continuum SPζ which is at $\frac{d}{2}$.

We can test this for the fuzzy sphere, the SPζ can be calculated as

$$\zeta_{D_n^2}(s) = 2n(n^2)^{-s} + 4H_{n-1}^{(2s-1)} \tag{18}$$

with H_n the Harmonic numbers [29],

$$H_n^r = \sum_{k=1}^{n} \frac{1}{k^r} \ . \tag{19}$$

At $s = 1$ this reduces to

$$\zeta_D(2) = 2n^{-2} + 4H_{n-1} \ , \tag{20}$$

which in the limit $n \to \infty$ behaves as $\log n + \gamma + O(1/n)$.

In the left hand plot of figure 2 ww show the spectral zeta function for the fuzzy sphere divided by $\log N$. The point of intersection is where we expect to read of the dimension, so to make this move visible we plot the difference between consecutive N values in the right hand plot. The zeroes of this can then be used as dimension estimators.

Fuzzy Torus

The fuzzy torus setup is very similar to that of the fuzzy sphere, except the Dirac operator is different. The Dirac operator is a combination of gamma matrices and commutators of the 'clock', C, and 'shift' operators, S which satisfy the relation $CS = qSC$, where $q^N = 1$ for some natural number N. The spectrum of the operator is a complicated expression involing the use of so-called 'q-numbers', all details for the fuzzy torus will be presented in [18]. The important result involving the fuzzy torus is that the above method still produces reasonable results for the dimension, shown in Figure 3

To understand why this is particularly interesting we need to explore the formalism in which all of the above theory for manifolds was built in.

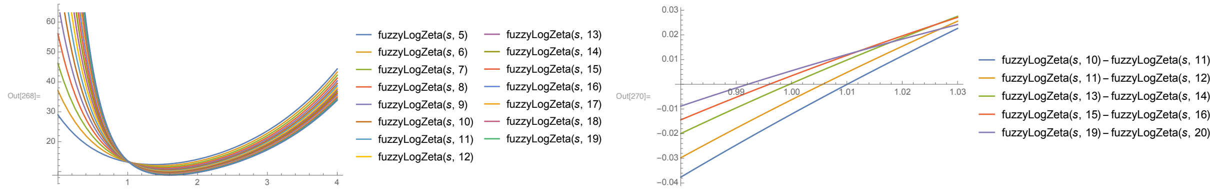


Figure 3: The SPζ for the fuzzy Torus for different n.

5.1.1 Volumes

Using the Dixmier trace defined in Appendix D we can see that the value of the SPζ rescaled by $\log(N)$ is precisely one of the terms in $\mathrm{Tr}_\omega(D^{-2})$. For the fuzzy sphere, the terms a_N are just finite selection of the continuum spheres infinite series. However, the situation for the fuzzy torus is different Which using the famous result of Eqn (35) we can relate the volume via Eq (14). The volumes of the fuzzy sphere are shown in Table 1.

The same process is done for the fuzzy torus and the volumes are shown in FigureZ4.The reason the fuzzy torus is particularly interesting is because the spectrum of Dirac operator is distinctly different from that of the continuum torus. It is not just a subset of the continuum like in the sphere case but a seemingly unrelated sequence a_N as we increase the matrix size in the fuzzy torus spectral triple. Which agrees with the Torus spectrum in the continuum limit, so the Dixmier trace, will agree.

We have also applied this to the random spectral triples generated in [17] which is currently being constructed in a paper with Lisa Glaser and John Barrett. Which makes use of the spectral dimension and a new object called the spectral variance.

It is immediately clear that this definition of the dimension only works if we can examine the same type of fuzzy space at different matrix sizes. While this is a disadvantage in so far as it requires more computer power to examine, it is nice in so far as it only recovers manifold dimension in the large matrix limit, hence giving a natural flow towards continuum physics. The techniques sued to examine the random geometries do not require a family of spectral triples, but what we can determine from such techniques is less that the SPζ.

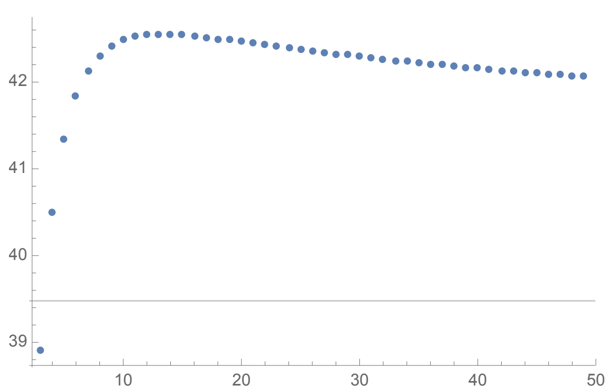


Figure 4: Volume of the fuzzy torus vs the Flat Torus. Solid line = Flat Torus, data points = volume of fuzzy torus.

5.2 Random Geometries

Barrett and Glaser [17] made use of the general form of a Dirac operator based on a matrix geometry spectral triple. The formula was presented in [28] and the setup goes as follows

5.2.1 Fuzzy spectral triples

To successfully describe a fuzzy spectral triple we have to deal with Clifford algebras over $\mathbb{R}^n$. You can view these as the tensor algebra generated by a set of elements in $\mathbb{R}^n$, $\{\gamma_a\}$ which satisfy the relation:

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$$

where η is a diagonal matrix with ± 1 in its entries. However, as we still require a Hilbert space in our spectral triple, which in turn requires us to have a positive definite inner product. We require that all the γ_a's are unitary and we have the standard hermitian inner product $(u, v) = \sum_i \bar{u}_i v_i$ on the Hilbert space. This requirement doesn't impose any restrictions on the γ_a as they form a finite group and finite groups always have a unitary representation [30]. Which just leaves us to choose a basis of the vector space these elements act on, V, which will be the Hilbert space of our spectral triple, such that the hermitian form is the standard one. Thus, if $\gamma_a^2 = 1$ then γ_a is Hermitian and if $\gamma_b^2 = -1$ then γ_b is anti-Hermitian. Note that if η has p entries of $+1$ and q entries of -1, then we say the Clifford module is of type (p, q), and the number $s = p - q \pmod 8$, called the signature of the clifford module, determines much of the characteristics of the Clifford module. The signature, s, also coincides with the KO-dimension of the associated spectral triple. Representing this Clifford algebra as complex matrices[10], which act on a vector space V, provides us with the Clifford module we require to define a fuzzy space.

Now let us define a type $(0, 0)$ matrix geometry:

Definition 1. A type $(0, 0)$ matrix geometry[11] is a real spectral triple of KO dimension $s_0 = 0$ and the following objects: $(\mathcal{H}_0, \mathcal{A}_0, D_0 = 0; \gamma_0 = 1, J_0)$

It can be shown (see [28]) that all type $(0, 0)$ matrix geometries are isomorphic to the case when $\mathcal{H}_0$ is a $\mathbb{C}$-linear vector subspace of $M_n(\mathbb{C})$ such that if $\mathbb{C}^n$ with its standard Hermitian inner product is a faithful left module of the algebra $\mathcal{A}_0$, we have that $am \in \mathcal{H}_0$ and also $m^* \in \mathcal{H}_0$. If this is the case then the representation of $\mathcal{A}_0$ on to $\mathcal{H}_0$ is just matrix multiplication, the real structure

[10] A Clifford module can be viewed as just representing a Clifford algebra as matrices. Hence the colloquial term for the γ_a discussed above as *gamma matrices*.

[11] The term matrix geometries arises from examples of this definiton being made by matrix contstructions. To see examples of how to construct some examples, see [28].

Table 1: The residue of the spectral zeta function at $s = \frac{d}{2}$ for the fuzzy spheres of different algebra sizes. Here n is the matrix algebra dimensions and N is the number of eigenvalues of the corresponding Dirac Operator

	KO Dimension						
n	10	100	200	300	455	1000	∞
$\frac{\zeta_{D_n^2}(1)}{\log(N)}$	3.84409	3.91247	3.9226	3.92751	3.93194	3.93899	4
Volume	1.922045	1.956235	1.9613	1.963755	1.96597	2	

is just Hermitian conjugation $J_0(\cdot) = (\cdot)^*$ and the inner product is just $(m_1, m_2) = \text{Tr}(m_1^* m_2)$. It can be shown that all the axioms for a type $(0,0)$ matrix geometry are satisfied by this collection of objects (see [28]).

We are now ready to define a fuzzy geometry. First we take all of the algebras in question to be *simple algebras*. Thats means $\mathcal{A} = M_n(\mathbb{C}), M_n(\mathbb{R})$ or $M_{n/2}(\mathbb{H})$. For the sake of brevity, the assumption that $\mathcal{A} = M_N(\mathbb{C})$ will be taken. Furthermore we impose that $\mathcal{H}_0 = M_n(\mathbb{C})$, where in the case of $\mathcal{A} = M_{n/2}(\mathbb{H})$ we express the quaternions as 2×2 complex matrices, such that $M_{n/2}(\mathbb{H}) \subset M_n(\mathbb{C})$

Definition 2. Let V be a Clifford module of type (p,q) with chirality operator γ and real structure C. For $p+q$ even let V be irreducible and for $p+q$ odd let the chiral subspaces $V_\pm$ be irreducible. A *fuzzy space* is a real spectral triple with the following objects

- $\mathcal{H} = V \otimes M_n(\mathbb{C})$
- $\Gamma = \gamma \otimes 1$
- $\mathcal{A} = M_n(\mathbb{C})$
- $J = C \otimes J_0$

where the inner product on $\mathcal{H}$ is defined by:

$$\langle v_1 \otimes m_1, v_2 \otimes m_2 \rangle = (v_1, v_2)\text{Tr}(m_1^* m_2)$$

and the action of the algebra is just my multiplication on $M_n(\mathbb{C})$, i.e. $\rho(a)(v \otimes m) = v \otimes (am)$. The actions of Γ and J on an element of $\mathcal{H}$ are as follows:

$$\Gamma(v \otimes m) = \gamma v \otimes m, \qquad J(v \otimes m) = Cv \otimes m^*$$

And the final object is the Dirac operator, D, where it can be shown (see [28]) that depending on the sign of ϵ', takes the following forms:

<u>$\epsilon' = 1$</u>

For this case we have that J and D commute and that $\langle D(v_1 \otimes m_1), v_2 \otimes m_2 \rangle = \langle v_1 \otimes m_1, D(v_2 \otimes m_2) \rangle$, we have that the Dirac operator has to be of the form [28]:

$$D(v \otimes m) = \sum_i \alpha^i v \otimes [L_i, m] + \sum_j \tau^j v \otimes \{H_j, m\} \tag{21}$$

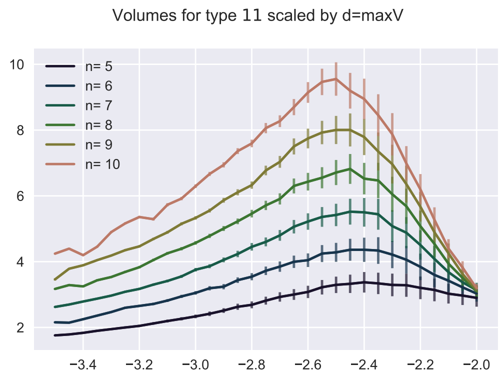
where α^i are products of gamma matrices and both α^i and L_i are anti-Hermitian matrices, and where τ^j are products of gamma matrices and both τ^j and H_j are Hermitian matrices.

<u>$\epsilon' = -1$</u>

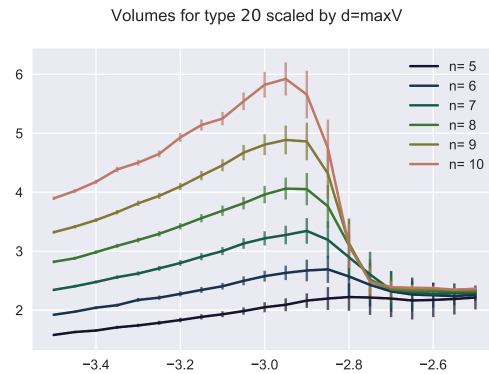
For this case, we have that J and D now anti-commute and also note that C anti-commutes with the γ_a in such cases, so we need to split the sums into sums where we have a product of even or odd number of γ_a. However we still require D to be self-adjoint, so we still require D to have either entirely Hermitian or entirely anti-Hermitian entries. This leaves us with the following form [28]:

$$\begin{aligned} D(v \otimes m) = &\sum_i \alpha_-^i v \otimes [L_i, m] + \sum_j \tau_-^j v \otimes \{H_j, m\} \\ &+ \sum_k \alpha_+^k v \otimes \{L_k, m\} + \sum_l \tau_+^l v \otimes [H_l, m] \end{aligned} \tag{22}$$

where the $+$ or $-$ subscript on α, τ indicates whether they are the product of an even number of γ_a, indicated by $+$, or by an odd number of γ_a, indicated by a $-$.



(a) Type 11 d=maxV



(b) Type 20 d=maxV

Figure 5: Volume of type 11 and type 20 with dimension determined by the maximum of the spectral variance. The phase transition value for type 11 is around 2.3 and for type 20 is around 2.7

This construction lends itself to Monte Carlo simulation [17], where the entries L, H in the Dirac operator are randomly generated, and the partition function is $Z = \int_{\mathcal{G}} S(D)\mathrm{d}D$. Where the action $S(D)$ that we will be using at is of the form $S(D) = g_4 D^4 + g_2 D^2$, which under the scaling of the Dirac operator by some constant, can rescaled $g_4 = 1$ (assuming $g_4 \neq 0$ to begin with). A phase transition was found in [17] and it is claimed that it is at this phase transition that manifold behaviour is expected to be observed. So some spectral measurements like those above for the sphere and torus were tested.

Volumes for the different types presented in [17] have been calculated, however there is a choice here of scaling as the simulation restricts the eigenvalue range to a fix region about the origin, with bigger matrix geometries just more densely packing the region. This artificial restriction is to do with the rescaling of the g_4 value. However, we can use the spectral dimension/variance to determine the dimension of each random geometry. Then we can use this to scale the eigenvalues to be $N^{1/d}$, where N is the number of eigenvalues. This is the feature that the SPζ relies on. Testing with integer values of d were tested, however, these are less desirable as the require *ad-hoc* assumptions. The volumes of this geometries are should for the spectral variance scaling are shown in Figures 5-6.

The volumes highlight the fact there are distinct behaviours that these geometries exhibit depending on the value of g_2, and motivate further research into these geometries.

A A brief introduction to spectral triples

Definition 3. A *spectral triple* is a triple $(\mathcal{H}, \mathcal{A}, D)$, where:

- $\mathcal{H}$ is a Hilbert space
- $\mathcal{A}$ is a $*$-unital algebra represented as bounded operators on $\mathcal{H}$.
- D is a self-adjoint operator on $\mathcal{H}$ such that the resolvent $(i + D)^{-1}$ is a compact operator and $[D, a]$ is bounded for each $a \in \mathcal{A}$.

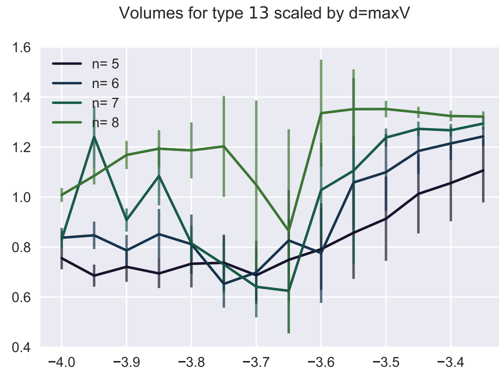


Figure 6: Volume of type 13 with dimension determined by the maximum of the spectral variance. The phase transition value for type 13 is around 3.7

Table 2: The KO dimension, n, of a real spectral triple is determined by the signs $\epsilon, \epsilon', \epsilon''$.

	KO Dimension							
n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1	1	-1	1	1	1	-1	1

We often require additional structure to our noncommutative geometries if we require them to be appropriate generalisations of Riemannian spin geometries. The first is a $\mathbb{Z}_2$-grading γ on the Hilbert space $\mathcal{H}$ such that:

$$\gamma a = a\gamma \quad (\forall a \in \mathcal{A}), \qquad \gamma D = -D\gamma$$

If such a γ exists then we the spectral triple is said to be *even*. The second is an anti-linear map $J\colon \mathcal{H} \to \mathcal{H}$ such that:

$$J^2 = \epsilon, \qquad JD = \epsilon' DJ, \qquad J\gamma = \epsilon''\gamma J \quad \text{(when } \gamma \text{ exists)}$$

where $\epsilon, \epsilon', \epsilon''$ take values from Table 2. We also require If such a J exists then the spectral triple is said to be *real* and J is referred to the as the *real structure*. The values for $\epsilon, \epsilon', \epsilon''$ are determined by the *KO dimension* of the real spectral triple, which for the spectral triple of a Riemannian spin manifold will be equal to the dimension of the manifold modulo 8.

B The Fuzzy Sphere

The fuzzy sphere is described by the spectral triple, $(M_n(\mathbb{C}), V \otimes M_n(\mathbb{C}), D_{FS})$. Where the Dirac operator, D_{FS} is given by

$$D = \gamma_0 + \sum_{i<j=1}^{3} \gamma_0\gamma_i\gamma_j \otimes [L_{ij}, \cdot] \tag{23}$$

where L_{ij} are n-dimensional representations of the generators for the Lie algebra $\mathfrak{so}(3)$ which satisfy $L_{ij} = -L_{jk}$ and

$$[L_{ij}, L_{kl}] = \delta_{jk}L_{il} + \delta_{il}L_{jk} - \delta_{jl}L_{ik} - \delta_{ik}L_{jl} \tag{24}$$

with the fact that $L_{ij} = -L_{jk}$. This form of the fuzzy sphere is given by Barrett [28], which differs slightly from the likes of Madore [4] in that its defined as a spectral triple whereas Madore does not present the Dirac operator. And it differs from Grosse-Prešnajder [31] in that the the K0-dimension of the spectral triple is equal to 2, whereas G-P K0-dimension is equal to 3. And in the continuum limit ($n \to \infty$) is it though that the K0-dimension should match that of the normal sphere which is equal to 2. It is possible to arrive back at the Grosse-Prešnajder Dirac operator as done by Barrett in [28], and for reference later on the form of the Grosse-Prešnajder Dirac operator is:

$$D_{\text{G-P}} = 1 + \sum_{i<j=1}^{3} \sigma_i\sigma_j \otimes [L_{ij}, \cdot] \tag{25}$$

The Dirac operators both yields the spectrum

$$Spec(D) = \pm 1, \pm 2, \pm 3, \cdots \pm n-1, +n \tag{26}$$

with the difference taking form in a disagreement of multiplicities of the values[12]. So as we let the matrix size tend to infinity we retrieve the spectrum $\pm\mathbb{Z}$, which is the spectrum for the Dirac operator on the commutative 2-sphere.

C Pseudodifferential Operators

Pseudo-differential operators are generalisations of differential operators in a specific way. So first lets define a differential operator. Note that there are equivalent definitions of differential and pseudodifferential operators that are coordinate free which lends themselves to use in noncommutative geometry. Such definitions require the use of jets and jet bundles, which are construction to allow a coordinate free expression of taylor series however for simplicity we will stick to definitions involving coordinates.

Definition 4. A differential operator, P, is given by a linear combination of derivatives given locally as

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{27}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in {\mathbb{Z}_+}^n$ is a multi-index. $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$

This acts on smooth functions with compact support in $\mathbb{R}^n$. However, by utilising Fourier transforms, one can view this as multiplication by

$$P(\xi) = \sum_{\alpha} a_\alpha \xi^\alpha \tag{28}$$

in Fourier space and then inverse Fourier transforming back. This can be expressed as

$$P(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi} P(\xi) u(y) \mathrm{d}y \mathrm{d}\xi. \tag{29}$$

We can now generalise to pseudo-differential operators

[12] The Barrett fuzzy sphere Dirac operator exhibits *fermion doubling*.

Definition 5. A pseudodifferential operator $P(x, D)$ on $\mathbb{R}^n$ is an operator whose value on the function $u(x)$ is

$$P(x, D)u(x) = \frac{1}{(2\pi)^n} \int \mathbb{R}^n e^{ix\cdot\xi} P(x, \xi)\hat{u}(\xi)\mathrm{d}\xi \tag{30}$$

where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$ and $P(x, \xi)$ belongs to a certain *symbol class* $S^m_{1,0}$. This equates to $P(x, \xi)$ being infinitely differentiable on $\mathbb{R}^n \times \mathbb{R}^n$ and such that

$$|D^\alpha_\xi D^\beta_\xi P(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|} \tag{31}$$

for all $x, \xi \in \mathbb{R}^n$ and for all multi-indices α, β. $C_{\alpha,\beta}$ are some constants and m is just a real number.

We say then that the differential operator $P(x, D)$ belongs to the class $\Psi^m_{1,0}$

D Dixmier Trace

Definition 6. Let T be a compact linear operator on a Hilber space $\mathcal{H}$ such that the following norm is finite

$$\|T\|_{1,\infty} = \sup_N \frac{\sum_{i=1}^N \mu_i(T)}{\log(N)} < \infty \tag{32}$$

where $\mu_i(T)$ are the eigenvalues of T arranged in descreasing order. Let

$$a_N = \frac{\sum_{i=1}^N \mu_i(T)}{\log(N)} \tag{33}$$

then the Dixmier trace is

$$Tr_\omega(T) = \lim_\omega (a_N). \tag{34}$$

Where $\lim_\omega$ is a scaled invariant extension of the usual notion of a limit such that:

- $\lim_\omega(a_N) \geq 0$ if $a_n \geq 0$
- $\lim_\omega(a_N) = \lim(a_N)$ when it exists in the normal sense
- $\lim_\omega(a_1, a_1, a_2, a_2, \dots) = \lim_\omega(a_N)$

The famous result by Connes [32] which states that

$$Tr_\omega(T) = \lim_{s\to 1_+} (s - 1)\zeta_{T^{-1}}(s) \tag{35}$$

So substituting $T = D^{-2}$, yields the result that

$$\mathrm{Tr}_\omega(D^{-2}) = \lim_{s\to 1_+} (s - 1)\zeta_{D^2}(s) \tag{36}$$

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