Geometry of fuzzy spaces

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Why do we care about fuzzy spaces?

$(M, g, S) \longleftrightarrow (C^\infty(M), \Gamma(S), \not{D})$

compact Riemannian spin manifold; always a commutative algebra; spinors; Dirac operator

noncommutative geometry (NCG) := ? $\longleftarrow (\mathcal{A}, \mathcal{H}, D)$

noncommutative

Standard model

$\downarrow$

$\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

$\mathcal{H} = \mathbb{C}^{96}$

$\not{D}$ = matrix of fermion masses

Noncommutative spacetime?

$\downarrow$

$\mathcal{A} = M_n(\mathbb{C}) \quad \mathcal{H} = V \otimes M_n(\mathbb{C})$

$$D = \sum_i \alpha^i \otimes [L_i, \cdot] + \sum_j \tau^j \otimes \{H_j, \cdot\}$$

anti-Hermitian; Hermitian

simplest NCG $\longrightarrow$ fuzzy space

Fuzzy examples

Fuzzy Sphere

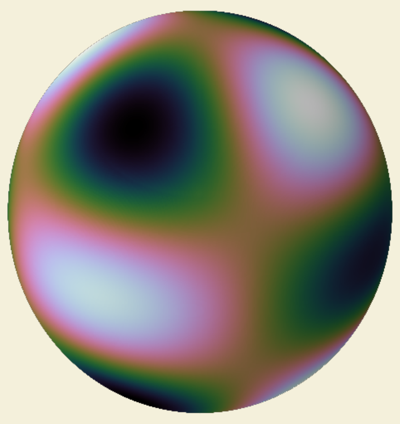
$$D = \gamma^0 + \sum_{i<j=1}^{3} \gamma^0 \gamma^i \gamma^j \otimes [L_{ij}, \cdot]$$

$\mathfrak{so}(3)$ generators

$$\mathcal{A}_n \simeq \bigoplus_{l=0}^{n-1} V_l$$

spanned by Y_m^l $\longrightarrow$ maximum energy/ minimum length

recover round metric on the sphere as $N \to \infty$



Random Fuzzy Spaces

- Using Monte Carlo methods to randomly generate the matrices L_i, H_j
- Probability distrubution given $e^{-S(D)}$ with $S(D) = Tr(D^4) + g_2 Tr(D^2)$
- $\exists$ phase transition as we vary g_2, what changes geometrically?!

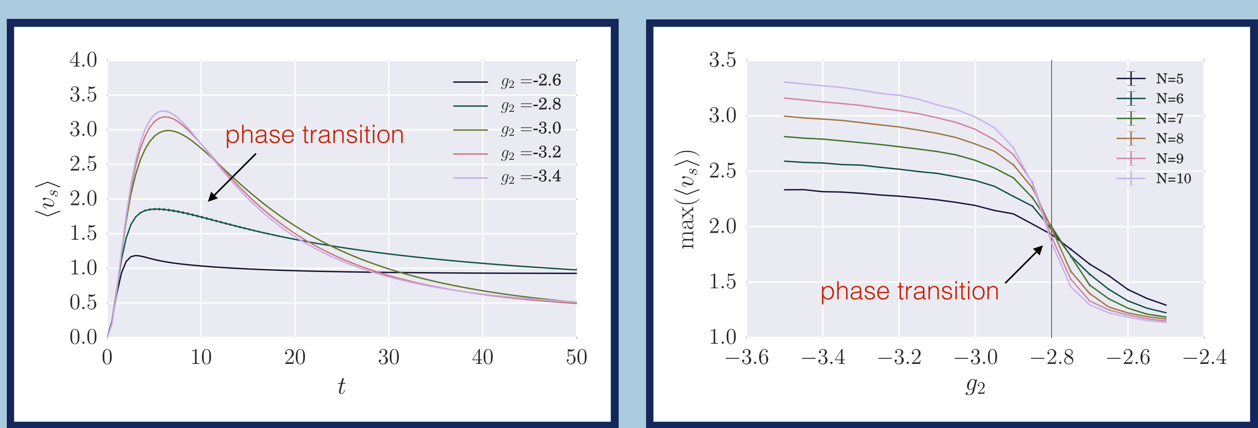
Fuzzy dimension

- In the literature the notion of spectral dimension is well studied.
- Defined using the Heat Kernel trace, $K_\Delta(t) = Tr(e^{-t\Delta}) = \sum_i e^{-t\lambda_i(\Delta)}$ as the following:

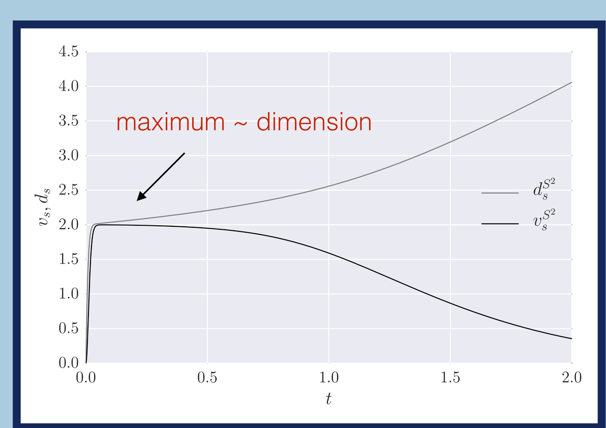
$$d_s(t) = -2t \frac{d \log(K(t))}{dt}$$

- Typically Dirac operators do not have zero eigenvalues and this dominates the series for large t.
- So we define the spectral variance as: $v_s(t) = d_s(t) - t \frac{dv_s(t)}{dt}$
- The maximum of the spectral variance, $\max(V_s)$, approximates the dimension

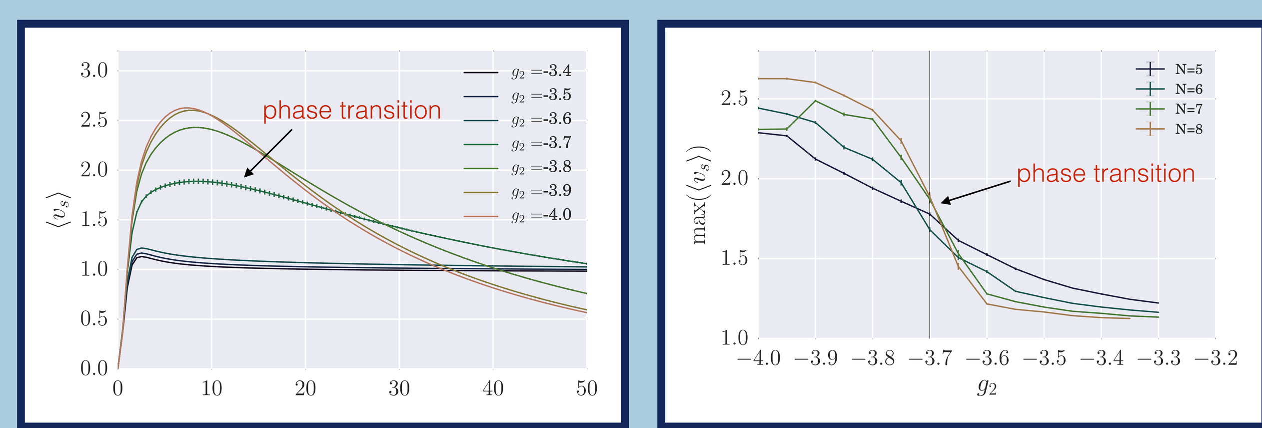
Type (2,0)



Fuzzy Sphere



Type (1,3)



Fuzzy volume

- Using heat kernel trace asymptotics we can geometric invariants

$$K_{D^2}(t) \overset{t\to 0}{=} t^{-\frac{d}{2}}(a_0 + a_2 t^2 + a_4 t^4 + \dots)$$

dimension

$$a_0 = \frac{Tr(Id)}{(4\pi)^{d/2}} Vol(M)$$

size of γ^i matrices

- Using the Mellin transform we can relate the heat kernel to the spectral zeta function

$$K_{D^2}(t) = \frac{1}{2\pi i} \oint ds\, t^{-s} \Gamma(s) \zeta_{D^2}(s)$$

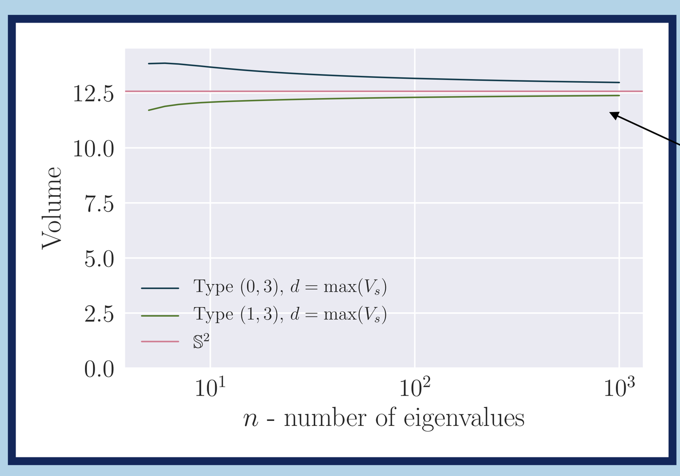
$$Vol(M) = \frac{(4\pi)^{d/2}}{Tr(Id)} \Gamma(s) Res_{s=\frac{d}{2}} \zeta_{D^2}(s)$$

- Making use of the Dixmier trace we can estimate the volumes

$$Res_{s=\frac{d}{2}} \zeta_{D^2}(s) = \lim_{N\to\infty} \frac{d}{2} \frac{1}{\log(N+1)} \sum_{n=0}^{N} \lambda_n (D^{-\frac{d}{2}})$$

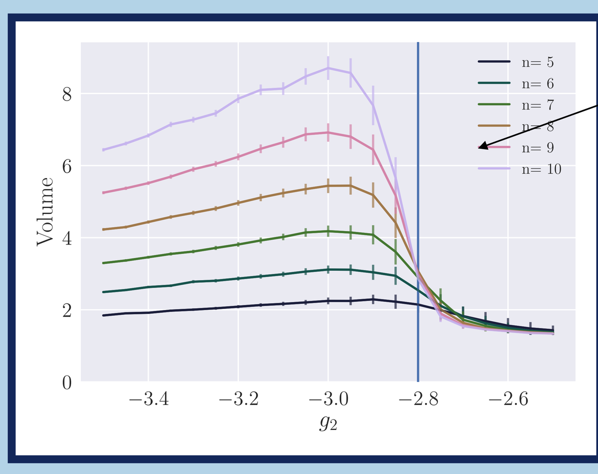
$$Vol_N(M) = \frac{d}{2} \frac{(4\pi)^{d/2}}{Tr(Id)} \frac{\Gamma(s)}{\log(N+1)} \sum_{i=0}^{N} \lambda_i (D^{-\frac{d}{2}})$$

Fuzzy Sphere



Volumes converge as matrix size grows

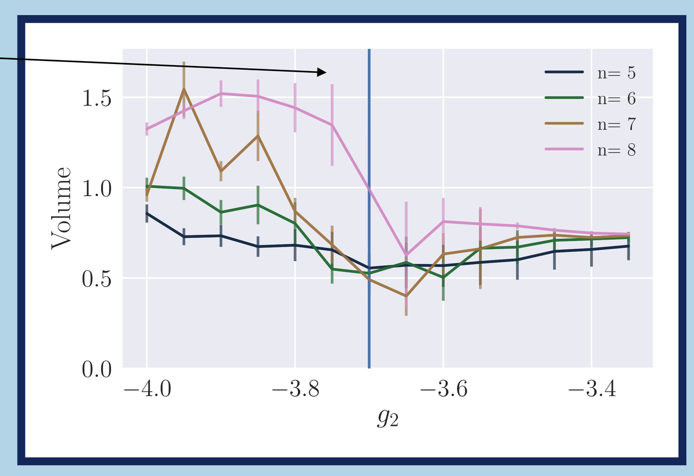
Type (2,0)



Volumes become size dependant after phase transition

At the phase transition we see matrix size independent features

Type (1,3)



Conclusion

- Fuzzy spaces exhibit non zero dimension and volumes
- Phase transition of random noncommutative geometries is linked to size independent geometry
- More fuzzy spaces need constructing and studying

References

[1] Barrett, John W. "Matrix geometries and fuzzy spaces as finite spectral triples." Journal of Mathematical Physics 56.8 (2015): 082301.
[2] Barrett, John W., and Lisa Glaser. "Monte Carlo simulations of random non-commutative geometries." Journal of Physics A: Mathematical and Theoretical 49.24 (2016): 245001.
[3] Glaser, Lisa. "Scaling behaviour in random non-commutative geometries." Journal of Physics A: Mathematical and Theoretical 50.27 (2017): 275201.