First Year Report - Noncommutative Geometry and Quantum Gravity

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1 Motivation

Noncommutative geometry[1] is an attempt to generalise the notion of differential geometry. The generalisation is taken by considering the fact that the differential nature of a manifold, M, is encoded in a commutative algebra of smooth functions $C^{\infty}(M)$. We then pose the question of what sort of structure has similar properties but where the corresponding algebra is noncommutative. This question has been extended to consider Riemannian spin manifolds and their appropriate generalisations with the aim of developing a good model for reality. The standard model of particle physics has a very natural origin within noncommutative geometry and current research is looking into incorporating gravity into the overall picture. To make sense of the basic objects in noncommutative geometry it is useful to cast the corresponding "normal" commutative picture into the correct framework. As the object we wish to generalise is that of a Riemannian spin manifold, below is a brief walk through the basics of spin geometry in the appropriate view point.

2 Introduction to Spin Geometry

Spin geometry has far reaching applications in mathematics and physics and is the topic on many books and papers. A very thorough and most famous is that by Lawson and Michelsohn [1] however for out applications it is very mathematically dense and there are more digestible books. For the more geometrically incline the book by Friedrich [2] is very good and for those who prefer to stay closely related to physics the books by Lawrie [3], Benn and Tucker [4] and specifically for those interested in noncommutative geometry and physics the book by Suijlekom [5] is excellent.

The basic object of interest are called Riemannian spin manifolds. These are manifolds with a riemannian metric and spin structure on them. T he formalism for spinors and therefore particle physics requires such an object, however, there is considerable prerequisite material needed in order to define a spin manifold. To begin with, we start with the definition of a Clifford algebra bundle[2]. Given a Riemannian manifold, M, we can define a Clifford algebra, $\mathrm{Cl}(T_xM, Q_g)$, at each point of the manifold, where T_xM is the tangent space at the point x and Q_g is a quadratic form induced by the metric g defined as follows:

$$Q_g(X_x) = g_x(X_x, X_x) \quad \forall X_x \in T_xM$$

by letting x vary we can construct the Clifford algebra bundle:

[1]Sometimes this is referred to as Noncommutative Differential geometry to distinguish it from Noncommutative Algebraic Geometry.

[2]For details on clifford algebras, see the Appendix A

Definition 1. The Clifford Algebra bundle $\mathrm{Cl}^+(TM)$ is the fibre bundle[3] over M, where the fibres are the clifford algebras $\mathrm{Cl}(T_xM, Q_g)$ and the transition functions are inherited from the tangent bundle TM, $t_{ij}\colon U_i \cap U_j \to SO(n)$, where $n = dim(M)$ and their action on $\mathrm{Cl}(T_xM, Q_g)$ is given by:

$$t_{ij}(v_1 v_2 \dots v_k) = t_{ij}(v_1) \dots t_{ij}(v_k)$$

Given a Clifford algebra bundle we can define the algebra of continuous real-valued sections denoted by $\mathrm{Cliff}^+(M) := \Gamma(\mathrm{Cl}^+(TM))$. By replacing Q_g with $-Q_g$ we can analogously define $\mathrm{Cl}^-(TM)$ and $\mathrm{Cliff}^-(M)$. We can also define:

$$\mathbb{C}\mathrm{liff}(M) := \mathrm{Cliff}^+(M) \otimes_{\mathbb{R}} \mathbb{C}$$

which is the space of continuous sections of the bundle of complexified algebras $\mathbb{C}\mathrm{l}(TM)$. Now we are ready to start describing what spin structures are and therefore what spinors are. As spinors are one of the most fundamental objects in quantum physics, a precise notion of what we mean is desirable.

Definition 2. A Riemannian manifold, M, is said to be *spin*^{c} if there exists a vector bundle $(S, M, End(S), f)$ such that there is an algebra bundle isomorphism:

$$\mathbb{C}\mathrm{l}(TM) \simeq End(S) \qquad M \text{ is even dimensional} \tag{1}$$
$$\mathbb{C}\mathrm{l}^0(TM) \simeq End(S) \qquad M \text{ is odd dimensional} \tag{2}$$

In such a case, the pair (M, S) is said to be a *spin*^{c} *structure*[4] for M.

If a spin^c structure (M, S) exists, the we refer to the bundle as the *spinor bundle* and the sections as *spinors*. However, not all of these spinors will be physical spinors. As we require physical spinors to be *square integrable* and so we now define the space of spinors we want.

Definition 3. The *Hilbert space of square-integrable spinors*, denoted $L^2(S)$, is define by the completion of $\Gamma(S)$ under the norm induced by the following inner product for $\phi_i \in \Gamma(S)$. :

$$(\phi_1, \phi_2)_M = \int_M \langle \phi_1, \phi_2 \rangle_S \sqrt{g}\, \mathrm{d}x$$

The inner product of spinors is just the clifford inner product at each point of the manifold, where the inner product on on the Clifford algebras can be defined on basis elements from its quadratic form as follows:

Definition 4. Let $\{e_i\}$ be an orthonormal basis[5] of V. Setting $\alpha = e_1 \dots e_p$ let $\hat{\alpha} = e_p \dots e_1$, then the inner product is

$$\langle e_{i_1} e_{i_2} \dots e_{i_p}, e_{j_1} e_{j_2} \dots e_{j_q} \rangle := \langle 1, e_{i_p} \dots e_{i_2} e_{i_1} e_{j_1} e_{j_2} \dots e_{j_q} \rangle = \begin{cases} 0 & \text{if } p \neq q \\ 0 & \text{if } e_{i_k} \neq e_{j_k} \text{ for any } k \\ Q(e_{i_1}) \dots Q(e_{i_p}) & \text{otherwise} \end{cases} \tag{3}$$

and can be extended linearly to arbitrary elements of $Cl(V, Q)$ by $\langle \alpha, \beta \rangle = \langle 1, \hat{\alpha}\beta \rangle$

[3]For more details on the construction of fibre bundles see Appendix B.s

[4]This definition looks a little contrived without knowing the motivation for it, which arises when we consider the Euclidean space $\mathbb{R}^n$ with the standard metric, δ. It can be shown that we have $\mathbb{C}\mathrm{l}(\mathbb{R}^{2n}) \simeq M_n(\mathbb{C})$ and $\mathbb{C}\mathrm{l}^0(\mathbb{R}^{2n+1}) \simeq M_n(\mathbb{C})$. And this definition is the generalisation to arbitrary Riemannian manifolds.

[5]So that $Q(e_i) = \pm 1$ and $B(e_i, e_j) = 0$ if $i \neq j$. Where B is defined as in the Appendix A.

However, the model we have for reality requires us to be able to pair up each spinor with its anti-particle spinor. To be able to do this mathematically requires us to have not just a spin$^{\mathbb{C}}$ manifold but a *spin manifold*

Definition 5. A Riemannian spin^c manifold is called a *Riemannian spin manifold* if there exists an anti-unitary operator $J_M \colon \Gamma(S) \to \Gamma(S)$ such that:

1. J_M commutes with the action of real valued continuous functions on $\Gamma(S)$.

2. J_M commutes with $\text{Cliff}^-(M)$ in the even cases and with $\text{Cliff}^-(M)^0$ in the odd cases.

The pair (S, J_M) is referred to as a *spin structure* on M and J_M is referred to as the *charge conjugation* on M. Another important structure which is necessary for understanding the basic objects in noncommutative geometry is the notion of a *chirality operator*. Let $\{\gamma_a\}_{a=1}^n$ be the generators of $\text{Cliff}^+(M)\big|_U$ for some some open neighbourhood, U, of M, where $\{x_a\}_{a=1}^n$ are the local coordinates. The γ_a satisfy: $\gamma_a\gamma_b + \gamma_b\gamma_a = 2g(\partial_a, \partial_b)$ and if we choose an orthonormal basis for $\Gamma(TM)\big|_U$ then the γ_a satisfy the relation:

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}$$

.

Definition 6. The *chiratity operator* can be constructed from the γ_a as follows:

$$\gamma_M = (-i)^m \gamma_1 \ldots \gamma_n$$

where $m = n/2$ (when n is even) or $m = (n-1)/2$ (when n is odd).

2.1 The Dirac Operator

The Dirac operator can be thought of a "square root" of the Laplacian of a space and contains a lot of information. The Dirac operator describes the dynamics of spinors but also encodes the metric information. Thus if we are given the Dirac operator we can extract the full metric and thus know everything we need to about the space we are making calculations in. Do define it however, the notion of a spin connection is required. Given a Riemannian spin manifold M with spin structure (S, J_M), let $\{E_a\}$ be a local orthonormal basis for $TM\big|_U$ then $g(E_a, E_b) = \delta_{ab}$. Let θ^a be the duals to E_a, then the Levi-Civita connection in this basis acts on vectors and one forms in the following way:

$$\nabla E_a = \tilde{\Gamma}_{a\ c}^{\ b} \, \mathrm{d}x^c \otimes E_b \tag{4}$$

$$\nabla \theta^a = -\tilde{\Gamma}^a{}_{bc} \, \mathrm{d}x^b \otimes \theta^c \tag{5}$$

And recall that if we have an orthonormal basis for $TM|_U$ then we have that $\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}$.

Definition 7. The spin connection ∇^S on the Spinor bundle $(S, M, End(S), f)$ is the lift of the Levi-Civita connection and in locally is given by:

$$\nabla_a^S \psi(x) = \left(\partial_a - \frac{1}{4} \tilde{\Gamma}^b{}_{ac} \gamma^d \gamma_d \right) \psi(x)$$

The notion of Clifford multiplication is necessary to define the Dirac operator so we include its definition in this setting below:

Definition 8. *Clifford multiplication* is defined as the linear map:

$$c\colon \Omega^1_{\mathrm{dR}}(M) \times \Gamma(S) \to \Gamma(S) \tag{6}$$

$$(\omega, \psi) \to \omega^{\#} \cdot \psi \tag{7}$$

where $\Omega^1_{\mathrm{dR}}(M)$ is the space of 1-forms on M and $\omega^{\#}$ vector field in $\Gamma(TM)$ associated to ω. The vector field acts an endomorphism on the fibres of S via the embedding $\Gamma(TM) \to \mathrm{Cliff}^+(M) \subset \Gamma(\mathrm{End}(S))$. Choosing local coordinates for $U \subset M$ we can write $\omega|_U = \omega_a \, \mathrm{d}x^a$ and we can write the Clifford multiplication as follows

$$c(\omega, \psi)|_U = \omega_a (\gamma^a \psi)|_U$$

where the $\mathrm{d}x^a$ have been identified with ∂_a via the metric and then embedded in $\mathrm{Cliff}^+(M)$ and become the γ^a. The pair $(\Gamma(S), c)$ is a *Clifford module* and will play an important roll in the noncommutative setting. For completeness the definition for a Clifford module has been included below:

Definition 9. A *Clifford module* over a compact Riemannian manifold (M, g) is a pair $(\Gamma(E), c)$ where $\Gamma(E)$ is the sections of a smooth vector bundle, E, and c is a $\mathrm{Cl}(TM)$-module homomorphism from $\Gamma(\mathrm{Cl}(TM)) \to \Gamma(\mathrm{End}(\Gamma(E)))$

We are now ready to define the Dirac operator for a Riemannian manifold, I have included its expression in local coordinates to indicate why so much preparatory work was necessary.

Definition 10. The *Dirac operator* for a spin manifold M with spin structure (S, J_M) is the composition of spin connection with Clifford multiplication and can be expressed in local coordinates as:

$$\not{D}_M = c \circ \nabla^S \colon \Gamma(S) \to \tag{8}$$

$$\not{D}_M \psi(x) = -i\gamma^a \left(\partial_a - \frac{1}{4} \tilde{\Gamma}^b{}_{ac} \gamma^c \gamma_b \right) \psi(x) \tag{9}$$

To see how we can unwrap the Dirac operator and retrieve the metric it is useful to look at the definition of a Dirac operator in terms of vielbeins.

Vielbeins are defined as follows: Take an orthonormal basis $\{e_a\}$ for $\Gamma(TM)$ such that[6] $g(e_a, e_b)(x) = \delta_{ab}$, we can express this basis in terms of the coordinate basis $\{\partial_\mu\}$ as follows $e_a(x) = e^\mu{}_a(x)\partial_\mu(x)$. A vielbein is defined to be $e^\mu{}_a$; the invertible transformation matrix, however the name often extends to the orthonormal basis also. Also note that the metric equation about can now be rewritten as $g_{\mu\nu}(x) e^\mu{}_a(x) e^\nu{}_b(x) = \delta_{ab}$ or equivalently

$$g_{\mu\nu}(x) = e_\mu{}^a(x) e_\nu{}^b(x) \delta_{ab} \tag{10}$$

where $e^\mu{}_a(x) e^a{}_\nu(x) = \delta^\mu{}_\nu$ and $e^\mu{}_a(x) e^b{}_\mu(x) = \delta_a{}^b$. The Dirac operator in terms of these vielbeins is defined as below:

$$\not{D}_M = -i\gamma^a e_a{}^b \nabla^S_b \tag{11}$$

So given a Dirac operator we can extract the vielbeins and therefore we can reconstruct the full metric.

[6] This is for Riemannian metrics, for pseudo-Riemannian metrics the δ in the relation is replaced with the corresponding pseudo-Riemannian equivalent.

Table 1: The KO dimension, n, of a real spectral triple is determined by the signs $\epsilon, \epsilon', \epsilon''$.

	KO Dimension							
n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1	1	-1	1	1	1	-1	1

3 Noncommutative Geometry

We are now ready to delve into noncommutative geometry and we start with a definition which took many years to refine and is still only appropriate for Riemannian spin geometries , although current research is investigating formalisms for "noncommutative psudeo-Riemannian spin geometry". A noncommutative geometry is described by a collection of objects called a spectral triple:

Definition 11. A *spectral triple* is a triple $(\mathcal{H}, \mathcal{A}, D)$, where:

- $\mathcal{H}$ is a Hilbert space
- $\mathcal{A}$ is a $*$-unital algebra represented as bounded operators on $\mathcal{H}$.
- D is a self-adjoint operator on $\mathcal{H}$ such that the resolvent $(i + D)^{-1}$ is a compact operator and $[D, a]$ is bounded for each $a \in \mathcal{A}$.

We often require additional structure to our noncommutative geometries if we require them to be appropriate generalisations of Riemannian spin geometries. The first is a $\mathbb{Z}_2$-grading γ on the Hilbert space $\mathcal{H}$ such that:

$$\gamma a = a\gamma \quad (\forall a \in \mathcal{A}), \qquad \gamma D = -D\gamma$$

If such a γ exists then we the spectral triple is said to be *even*. The second is an anti-linear map $J \colon \mathcal{H} \to \mathcal{H}$ such that:

$$J^2 = \epsilon, \qquad JD = \epsilon' DJ, \qquad J\gamma = \epsilon''\gamma J \quad \text{(when } \gamma \text{ exists)}$$

where $\epsilon, \epsilon', \epsilon''$ take values from Table 1. We also require If such a J exists then the spectral triple is said to be *real* and J is referred to the as the *real structure*. The values for $\epsilon, \epsilon', \epsilon''$ are determined by the *KO dimension* of the real spectral triple, which for the spectral triple of a Riemannian spin manifold will be equal to the dimension of the manifold modulo 8.

We also impose to two extra conditions which are chosen to keep up from straying too far away from commutative geometry. Firstly, using the J operator we can define the *opposite algebra* which entails the right action of the algebra. If $a \in \mathcal{A}$ then we can define $a^\circ = Ja^*J^{-1}$. Thus we can define the right action of the algebra on the Hilbert space to be $\psi b = b^\circ \psi$. The first requirement is that:

$$[a, b^\circ] = 0 \qquad \textit{Order zero condition}$$

which is used to model the fact that if we take a function $f \in C^\infty(M)$ for some Riemannian spin manifold and ψ some spinor, then $(f\psi)(x) = f(x)\psi(x) = (\psi f)(x)$. In other words, in commutative geometry we cannot distinguish between left and right actions of the algebra of functions and we want to keep some property of this commutative geometry feature even when the 'functions' no longer commute. The other condition is to do with the fact that commutation with the Dirac operator should also commute with the right action:

$$[[D, a]\, , b^\circ] = 0 \qquad \textit{First order condition.}$$

If we again look at commutative geometry, the Dirac operator can be expressed as in Eq (11), and thus: $[D, f]\,\psi = -i\gamma^a e^b_a(\nabla^S_b f)\psi$ and so acts as just multiplying pointwise by a function and so will commute with a right multiplication by a function. It is this property we wish to persist in our noncommutative geometry. We also note that there are noncommutative geometry models where higher order conditions are available, notably the work by Boyle and Farnsworth [6].

We are now ready to write down the spectral triple for a commutative geometry (a Riemannian spin manifold):

Definition 12. The real spectral triple for a Riemannian spin manifold, M, with spin structure (S, J_M) is the structure $\left(L^2(S), C^\infty(M), \not{D}_M; \gamma_M, J_M\right)$, where $L^2(S)$ is the Hilbert space as the space of square integrable spinors and $C^\infty(M)$ is the algebra of smooth functions. And where $\not{D}_M$, γ_M and J_M are the Dirac operator, chirality operator and charge conjugation as defined in the previous section.

4 Fuzzy Space

We are going to look at a specific type of noncommutative geometry, referred to as a *fuzzy space*. Fuzzy spaces belong to a class of noncommutative geometries called *matrix geometries*, which briefly are a spectral triple that is a product of a type $(0, 0)$ matrix geometry with a Clifford module. I will go onto define this is more detail below. Fuzzy spaces can be thought of as approximating Riemannian spin manifold with $n \times n$ matrices where n is an integer free to choose. Firstly, some important facts about Clifford modules have been described:

To successfully describe a fuzzy space we have to deal with Clifford algebras over $\mathbb{R}^n$ such that $\mathrm{Cl}(\mathbb{R}^n)$ has generators $\{\gamma_a\}^n_{a=1}$which satisfy:

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$$

where η is a diagonal matrix with ±1 in its entries. However, as we still require a Hilbert space in our spectral triple, which in turn requires us to have a positive definite inner product. We require that all the γ_a's are unitary and we have the standard hermitian inner product $(u, v) = \sum_i \bar{u}_i v_i$ on the Hilbert space. This requirement doesn't impose any restrictions on the γ_a as they form a finite group and finite groups always have a unitary representation [7]. Which just leaves us to choose a basis of the vector space these elements act on, V, which will be the Hilbert space of our spectral triple, such that the hermitian form is the standard one. Thus, if $\gamma_a^2 = 1$ then γ_a is Hermitian and if $\gamma_b^2 = -1$ then γ_b is anti-Hermitian. Note that if η has p entries of $+1$ and q entries of -1, then we say the Clifford module is of type (p, q), and the number $s = p - q \pmod 8$, called the signature of the clifford module, determines much of the characteristics of the Clifford module. The signature, s, also coincides with the KO-dimension of the associated spectral triple. Representing this Clifford algebra as complex matrices[7], which act on a vector space V, provides us with the Clifford module we require to define a fuzzy space.

Now let us define a type $(0, 0)$ matrix geometry:

Definition 13. A type $(0, 0)$ matrix geometry[8] is a real spectral triple of KO dimension $s_0 = 0$ and the following objects: $(\mathcal{H}_0, \mathcal{A}_0, D_0 = 0; \gamma_0 = 1, J_0)$

It can be shown (see [8]) that all type $(0, 0)$ matrix geometries are isomorphic to the case when $\mathcal{H}_0$ is a $\mathbb{C}$-linear vector subspace of $M_n(\mathbb{C})$ such that if $\mathbb{C}^n$ with its standard Hermitian inner product is a faithful left module of the algebra $\mathcal{A}_0$, we have that $am \in \mathcal{H}_0$ and also $m^* \in \mathcal{H}_0$. If this is the case then the representation of $\mathcal{A}_0$ on to $\mathcal{H}_0$ is just matrix multiplication, the real structure is just Hermitian

[7]A Clifford module can be viewed as just representing a Clifford algebra as matrices. Hence the colloquial term for the γ_a discussed above as *gamma matrices*.

[8]The term matrix geometries arises from examples of this definiton being made by matrix contstructions. To see examples of how to construct some examples, see [8].

conjugation $J_0(\cdot) = (\cdot)^*$ and the inner product is just $(m_1, m_2) = \mathrm{Tr}(m_1^* m_2)$. It can be shown that all the axioms for a type $(0,0)$ matrix geometry are satisfied by this collection of objects (see [8]).

We are now ready to define a fuzzy geometry. First we take all of the algebras in question to be *simple algebras*. Thats means $\mathcal{A} = M_n(\mathbb{C}), M_n(\mathbb{R})$ or $M_{n/2}(\mathbb{H})$. For the sake of brevity, the assumption that $\mathcal{A} = M_N(\mathbb{C})$ will be taken. Furthermore we impose that $\mathcal{H}_0 = M_n(\mathbb{C})$, where in the case of $\mathcal{A} = M_{n/2}(\mathbb{H})$ we express the quaternions as 2×2 complex matrices, such that $M_{n/2}(\mathbb{H}) \subset M_n(\mathbb{C})$

Definition 14. Let V be a Clifford module of type (p,q) with chirality operator γ and real structure C. For $p+q$ even let V be irreducible and for $p+q$ odd let the chiral subspaces $V_\pm$ be irreducible. A *fuzzy space* is a real spectral triple with the following objects

- $\mathcal{H} = V \otimes M_n(\mathbb{C})$
- $\Gamma = \gamma \otimes 1$
- $\mathcal{A} = M_n(\mathbb{C})$
- $J = C \otimes J_0$

where the inner product on $\mathcal{H}$ is defined by:

$$\langle v_1 \otimes m_1, v_2 \otimes m_2 \rangle = (v_1, v_2)\mathrm{Tr}(m_1^* m_2)$$

and the action of the algebra is just my multiplication on $M_n(\mathbb{C})$, i.e. $\rho(a)(v \otimes m) = v \otimes (am)$. The actions of Γ and J on an element of $\mathcal{H}$ are as follows:

$$\Gamma(v \otimes m) = \gamma v \otimes m, \qquad J(v \otimes m) = Cv \otimes m^*$$

And the final object is the Dirac operator, D, where it can be shown (see [8]) that depending on the sign of ϵ', takes the following forms:

$\underline{\epsilon' = 1}$

For this case we have that J and D commute and that $\langle D(v_1 \otimes m_1), v_2 \otimes m_2 \rangle = \langle v_1 \otimes m_1, D(v_2 \otimes m_2) \rangle$, we have that the Dirac operator has to be of the form [8]:

$$D(v \otimes m) = \sum_i \alpha^i v \otimes [L_i, m] + \sum_j \tau^j v \otimes \{H_j, m\} \tag{12}$$

where α^i are products of gamma matrices and both α^i and L_i are anti-Hermitian matrices, and where τ^j are products of gamma matrices and both τ^j and H_j are Hermitian matrices.

$\underline{\epsilon' = -1}$

For this case, we have that J and D now anti-commute and also note that C anti-commutes with the γ_a in such cases, so we need to split the sums into sums where we have a product of even or odd number of γ_a. However we still require D to be self-adjoint, so we still require D to have either entirely Hermitian or entirely anti-Hermitian entries. This leaves us with the following form [8]:

$$D(v \otimes m) = \sum_i \alpha_-^i v \otimes [L_i, m] + \sum_j \tau_-^j v \otimes \{H_j, m\} \tag{13}$$

$$+ \sum_k \alpha_+^k v \otimes \{L_k, m\} + \sum_l \tau_+^l v \otimes [H_l, m] \tag{14}$$

where the $+$ or $-$ subscript on α, τ indicates whether they are the product of an even number of γ_a, indicated by $+$, or by an odd number of γ_a, indicated by a $-$.

To show the usefulness of this rather abstract concept, a brief account of a well studied example is that of the *fuzzy sphere* is included below.

4.1 The Fuzzy Sphere

For an appropriate approximation to the sphere, which gets closer and closer to model a sphere as we increase the matrix size, n, we utilise the fact that the sphere has $SO(3)$ symmetry. If we take the Lie algebra of $SO(3)$, namely $so(3)$ and take an n-dimensional representation of it. Its generators are to be denoted by L_{ij}, with $i < j = 1, 2, 3$ such that they satisfy:

$$[L_{ij}, L_{kl}] = \delta_{jk} L_{il} + \delta_{il} L_{jk} - \delta_{jl} L_{ik} - \delta_{ik} L_{jl}$$

with the fact that $L_{ij} = -L_{jk}$. These generators, represented into the algebra, $M_n(\mathbb{C})$, with be the Hermitian/anti-Hermitian matrices that go into the Dirac operator. So the precise set up for the fuzzy sphere has undergone some changes in the recent past. One of the most noteworthy approaches is that by Grosse and Prešnajder [9], however they choose a type $(0, 3)$ fuzzy space, but this yields a KO dimension of $s = 3$ and as we want it to be the fuzzy analogue to the spheres dimensions which is $s = 2$. However, the method implemented by Barrett [8], enriches the $(0, 3)$ geometry to a $(1, 3)$ geometry, where the new Hermitian generator is denoted by γ_0. In which the KO dimension would be $s = 3 - 1 = 2$, as desired for a fuzzy sphere.

The Dirac operator for the fuzzy sphere is found to be:

$$D = \gamma_0 + \sum_{i<j=1}^{3} \gamma_0 \gamma_i \gamma_j \otimes [L_{ij}, \cdot] \tag{15}$$

we note that the product $\gamma_0 \gamma_i \gamma_j$ is anti-Hermitian and an odd product so this definition of the Dirac operator agrees with the Dirac operator definitions in (12) and (14). It is possible to arrive back at the Grosse-Prešnajder Dirac operator as done by Barrett in [8], and for reference later on the form of the Grosse-Prešnajder Dirac operator is:

$$D_{\text{G-P}} = 1 + \sum_{i<j=1}^{3} \sigma_i \sigma_j \otimes [L_{ij}, \cdot] \tag{16}$$

The Dirac operators both yields the spectrum

$$Spec(D) = \pm 1, \pm 2, \pm 3, \cdots \pm n - 1, +n \tag{17}$$

So as we let the matrix size tend to infinity we retrieve the spectrum $\pm\mathbb{Z}$, which is the spectrum for the Dirac operator on the commutative 2-sphere.

The link between these Dirac operators and the commutative 2-sphere's Dirac operator is more clearly seen when the commutative 2-spheres Dirac operator is expressed as embedded from $\mathbb{R}^3$ as opposed to intrinsically in terms of local coordinates. A method to get a general hyper-surface is reviewed below, and an attempt to look at the fuzzy analogue for the ellipsoid is pursued.

However, finally, the reason current research into noncommutative geometry as a model for quantum gravity is due to the interpretation the 'fuzziness' has on the small scale of the fuzzy sphere. A great paper for further details about the following is that by Madore [10]. In brief, it explains the reasons why noncommutativity is necessary for a finite model of a sphere, but that is still $SO(3)$ invariant. The premise is that if we consider the commutative 2-sphere, and look at the algebra of coordinates, $C(\mathbb{S}^2)$. This is the algebra spanned by the products of the coordinates x^1, x^2, x^3, so elements have polynomial expansions such as:

$$f(x^i) = f_0 + f_a x^a + f_{ab} x^a x^b \dots$$

for some constants $f_0, f_a, f_{ab}, \dots$ and where Einstein summation is assumed. This algebra has the ability to separate the points of a sphere and is also dense is the algebra of smooth functions $C^\infty(\mathbb{S}^2)$, so it is good enough to deal with just $C(\mathbb{S}^2)$ in this setting.

Now if we approximate the functions by truncating their series expansion after the n^{th} products, i.e. $f(x^i) = f_0 + f_a x^a + f_{ab} x^a x^b \cdots + f_{a_1 a_2 \ldots a_n} x^{a_1} x^{a_2} \ldots x^{a_n}$ and take all the functions of this type, we would end up with a n-dimensional vector space of the entries $f_0, \ldots, f_{a_1 a_2 \ldots a_n}$. In order to turn this vector space into an algebra requires some thought. The normal product of two elements f, g, we would obviously be taken out of this set of objects as it would require terms such as $x^{a_1} x^{a_2} \ldots x^{a_n} x^{a_{n+1}}$ etc. Looking at the case when $n = 1$, we could define a product to be $f \cdot g = f_0 g_0 + \sum_{a=1}^{3} f_a g_a x^a$, this would turn the vector space into an algebra. However, we can identify this algebra with four copies of $\mathbb{C}$, which is exactly the algebra of functions at four discrete points. This product can be extended to any value for n in the straight forward manner and we still only get an algebra which is the functions on a set of discrete points. This is unappealing because these point will not be invariant under the action of $SO(3)$ and thus we have no notion of a fuzzy sphere.

A way to preserve this invariance under the action of $SO(3)$ is to make the algebra we desire noncommutative. We start by taking the n-dimensional generators of $su(2) \simeq so(3)$, and denote them by L_a, the we set $x^a = \kappa L_a$, where κ is a constant set by requiring $\sum_i (x^i)^2 = 1$ (κ's value will vary with the value for n). Firstly we notice that the coordinates no longer commute, we have $[x^1, x^1] = i\kappa x^3$, and the cyclic permutations. So the notion of a point (x^1, x^2, x^3) vanishes as we can never know all three entries at the same time. However, if we take a look at the 2-dimensional representations; the Pauli matrices, we notice that we get 2 eigenvalues for each generator, namely ± 1. This is interpreted as being able to only distinguish which hemisphere of the sphere we lie in. As we increase the dimension of the representation we get more eigenvalues for each of the generators, which means we can narrow down the zone we lie. Which gives credence in the name fuzzy sphere as the spaces we end up with are like the points of a sphere have been smeared together in a certain fashion. Also as for a given n, the value of $\kappa = \frac{1}{n^2-1}$ we have that as $n \to \infty$ then $\kappa \to 0$, so the coordinates in the limit commute and we retrieve the normal commutative sphere. This interpretation gives us a natural picture of what we mean by a 'quantum geometry'. However, there is another interpretation which is arguably more enticing as it imposes a cutoff on the spherical harmonics and in turn the energy states. For a full treatment of this view point see [11].

Going back to the algebra of coordinates, we can decompose $C(\mathbb{S}^2)$ in to a direct sum of irreducible representations of $su(2)$:

$$C(\mathbb{S}^2) \simeq \bigoplus_{l=0}^{\infty} V_l$$

where V_l is the vector space underlying the irreducible representation of $su(2)$ with the highest weight $l \in \mathbb{N}$ which is spanned by the spherical harmonics $Y_{l,m}$. We can then impose a cutoff in the energy spectrum by ignore all but the first $n+1$ representations in the decomposition of $C(\mathbb{S}^2)$. Thus, in the fuzzy sphere's spectral triple we can take the *fuzzy spherical harmonics*[9] $\hat{Y}_{l,m}$ to be the generators of $M_{n+1}(\mathbb{C})$, where[10] $l < n$. We can then decompose the algebra into

$$\mathcal{A}_n \simeq \bigoplus_{l=0}^{n} V_l$$

Thus so a fuzzy sphere can be viewed as having an energy cutoff, which can be recast as a minimal renderable distance, i.e. a Planck length. The implications of a planck length being a natural outcome of requiring the underlying space to be noncommutative is a very appealing property and the idea is that a noncommutative analogue to a spacetime will provide a good model for quantum gravity.

[9]These are matrix versions of the spherical harmonics, there precise constructure can be found in [12].

[10]Note that the reason $l < n$ not $l < n + 1$ is because the index l starts at zero. So there are still $n + 1$ generators for $M_{n+1}(\mathbb{C})$

A brief note on the standard model in noncommutative geometry. This is possibly the most talked about triumph of noncommutative geometry is that the standard model presents itself as a noncommutative internal structure attached to a commutative Minkowski spacetime. There are many book walking through the details on how you formulate integrals and quantum field theory on these *almost-commutative manifolds*. The book by Suijlekom [5] is an excellent introduction to these topics however the book by Connes and Marcolli [13] is much more involved and the finer details and subtleties can be found there.

5 Present and Future Work

5.1 The Fuzzy Ellipsoid

Not many examples of fuzzy spaces exist that are deemed approximations of commutative manifolds, so an investigation into constructing a fuzzy ellipsoid was made. This would be a very useful fuzzy space as it would be the first non-symmetric fuzzy space and may provide vital insight into how to generalise the procedure of 'fuzzification' to other manifolds. Firstly, the Dirac operator for an ellipsoid was found as an embedding in $\mathbb{R}^3$, as was suggested by the fuzzy sphere example.

5.1.1 Dirac operators for Hypersurfaces of $\mathbb{R}^n$

The following method for getting the induced Dirac operator on a hypersurface is a very brief and to the point method taken from [14] and calculations for the sphere and ellipsoids are given. For full rigour of the results and steps to arrive at the formula, please refer to [14].

The method of constructing the Dirac operator begins with taking the polynomial that defines the hypersurface, $f(x^i) = 0$. First, for an embedding in $\mathbb{R}^n$ we take n Dirac matrices, $\{\gamma_a\}$ that satisfy the relation $\gamma_a\gamma_b + \gamma_b\gamma_a = -2\delta_{ab}$. Specifically, we have that $\gamma_a^2 = -1$. Then we find the unit normals to the surface $\boldsymbol{n}$. This can be done by using the formula $\boldsymbol{n} = \frac{\nabla f}{|\nabla f|}$. Then the formula for the Dirac operator, taken from [**?**], is as follows:

$$D = \sum_{i,j=1}^{3} (\gamma_i\gamma_j + \delta_{ij}) n_i \partial_j + \frac{1}{2}\mathrm{div}(n) \tag{18}$$

Where n_i are the ith components of the normal vector and $\mathrm{div}(n) = \sum\limits_{i,j=1}^{3} (\delta_{ij} - n_i n_j)\, \partial_i n_j$.

5.1.2 The Dirac operator for $\mathbb{S}^2$ embedded in $\mathbb{R}^n$

For the sphere defined as: $\sum\limits_{i=1}^{3} (x^i)^2 = 1$, we find the normals by the method outine above and get: $\boldsymbol{n}^i = x^i$. Thus we can calculate: We note that: $\partial_i n_j = \delta_{ij}$ and therefore:

$$\mathrm{div}(n) = \sum_{i,j=1}^{3} \delta_{ij}\delta_{ij} - n_i n_j \delta_{ij} = \underbrace{\sum_{i=1}^{3} \delta_{ii}}_{=3} - \underbrace{\sum_{i=1}^{3} x_i^2}_{=1} = 2 \tag{19}$$

Making use of this result in (18) we can rewrite the Dirac operator as:

$$D = \sum_{i,j=1}^{3} (\gamma_i\gamma_j + \delta_{ij}) x_i \partial_j + 1 = \sum_{i\neq j=1}^{3} \gamma_i\gamma_j x_i \partial_j + \sum_{i=1}^{3} (\gamma_i^2 + 1) x_i \partial_j + 1 \tag{20}$$

Here we use the fact that $\gamma_i^2 = -1$ and thus the middle term vanishes in general to give:

$$D = \sum_{i \neq j=1}^{3} \gamma_i \gamma_j x_i \partial_j + 1 \tag{21}$$

Now using that $\gamma_i \gamma_j = -\gamma_j \gamma_i$ we get the final form for the Dirac operator on $\mathbb{S}^2$:

$$D_{\mathbb{S}^2} = \sum_{i<j=1}^{3} \gamma_i \gamma_j (x_i \partial_j - x_j \partial_i) + 1 \tag{22}$$

Comparing this Dirac operator to the Dirac operators for the fuzzy sphere in Eq (15) and (16), we can clearly see a relation. The basic premise is that the vector fields $X_{ij} = x_i \partial_j - x_j \partial_i$ form are the *commutative limits* of the commutators with the generators of the Lie algebra, $[L_{ij}, \cdot]$. This interpretation is discussed in further detail in [8].

An attempt to extend this relationship to the case for an Ellipsoid was made as outlines below:

5.1.3 The Dirac operator on the Ellipsoid

Now for the Ellipsoid defined by: $\sum_{i=1}^{3} (\frac{x_i}{\alpha_i})^2 = 1$, the calculation follows the same steps however, the normals now have a more complicated form due the require them that have unit magnitude. The normals here are: $n_i = N \frac{x^i}{\alpha_i^2}$, where

$$\frac{1}{N} = \sqrt{\sum_{k=1}^{3} \frac{x_k^2}{\alpha_k^4}}$$

The complexities lie in the div(n) terms as that requires use to take derivatives of N, which brings up extra terms. However the first term in Eqn (18) can be computed in a similar fashion to sphere case and we arrive at:

$$D_{\mathbb{E}^2} = \sum_{i<j=1}^{3} \gamma_i \gamma_j N (\frac{x_i}{\alpha_i^2} \partial_j - \frac{x_j}{\alpha_j^2} \partial_i) + \frac{1}{2} \mathrm{div}(n) \tag{23}$$

The div(n) can also be computed, and is found to be:

$$\mathrm{div}(n) = N \sum_{j=1}^{3} \left(\frac{1}{\alpha_j^2} - N^2 \frac{x_j^2}{\alpha_j^6} \right) \tag{24}$$

So the total Dirac Operator for the Ellipsoid is:

$$D_{\mathbb{E}^2} = N \left(\sum_{i<j=1}^{3} \gamma_i \gamma_j (\frac{x_i}{\alpha_i^2} \partial_j - \frac{x_j}{\alpha_j^2} \partial_i) + \frac{1}{2} \sum_{j=1}^{3} \left(\frac{1}{\alpha_j^2} - N^2 \frac{x_j^2}{\alpha_j^6} \right) \right) \tag{25}$$

This is a much more complicated Dirac operator as the presence of N involves fractions with the coordinates in the denominator, which the 'fuzzy' analogue is currently unknown. I have included the calculation to explicitly show that this is the correct Dirac operator for an ellipsoid in Appendix C. Several attempts have been made to find make a sensible correspondence but with minimal success.

5.2 Future Work

The next port of call is to look at Dirac operators on general coadjoint orbits. As the sphere is a coadjoint orbit of $SU(2)$, the idea is to construct more examples using this correspondence as a guide line. More generally, investigations in to the *commutative limit* of a fuzzy space is to be investigated. However, more examples of fuzzy spaces is required to make fruitful progress in this topic. Also investigations into looking for the noncommutative analogue to a Lorentzian manifold is planned. Some headway has been made in this topic, see for example [?] [15] amongst other. However, there is still much to uncover in this topic.

A Algebra

Definition A.1. A vector space, V, over a field $\mathbb{F}$, is a set of objects that satisfy the following conditions:

- It is closed under finite vector addition: $\forall v, w \in V \Rightarrow v + w \in V$.
- It is closed under scalar multiplication: $\forall \lambda \in \mathbb{F}, v \in V \Rightarrow \lambda v \in V$

Definition A.2. If U, V are complex vector spaces the the map $f\colon U \to V$ is said to be *antilinear* if $f(u + v) = f(u) + f(v)$ and $f(\lambda u) = \bar{\lambda} f(u)$, where $\bar{\lambda}$ denotes complex conjugation.

Definition A.3. A *isometry* between normed vector spaces, U, V, is a linear map $f\colon U \to V$ that preserves the norm: $\|f(u)\| = \|u\|$ for all $u \in U$.

Definition A.4. A Hilbert space, $\mathcal{H}$ is a vector space with an inner product $\langle f, g\rangle$ such that the norm defined by

$$\|f\| = \sqrt{\langle f, f\rangle}$$

turns $\mathcal{H}$ into a complete metric space.

Definition A.5. An algebra $\mathcal{A}$ over field $\mathbb{F}$ is a vector space equipped with a bilinear associative product (multiplication). An algebra is said to be unital if it has an identity element with respects to the multiplication.

Definition A.6. Let R be a ring and 1_R be the multiplication identity. A *left R-module*, $\mathcal{M}$, consists of an abelian group $(\mathcal{M}, +)$ and an operation $\cdot : R \times \mathcal{M} \to \mathcal{M}$ such that $\forall r, s \in R$ and $\forall x, y \in \mathcal{M}$:

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(rs) \cdot x = r \cdot (s \cdot x)$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $1_R \cdot x = x$

The only difference between an algebra and a module is that an algebra is a module over a ring, where the ring is also a field.

Definition A.7. A $*$-algebra or *involutive algebra* is an algebra $\mathcal{A}$ together with a conjugate linear map called the involution: $*\colon \mathcal{A} \to \mathcal{A}$ such that $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for any $a, b \in \mathcal{A}$.

Definition A.8. A representation of a finite-dimensional $*$-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$. Where $\mathcal{H}$ is a Hilbert space and π is a $*$-algebra map:

$$\pi\colon \mathcal{A} \to L(\mathcal{H})$$

Definition A.9. Given a vector space V (over $\mathbb{F}$) and a quadratic form, Q on V. We define the *Clifford Algebra* $\mathrm{Cl}(V, Q)$ as the algebra generated (over $\mathbb{F}$) by the vectors $v \in V$ and the multiplicative unit 1_F such that:

$$v \cdot v = v^2 = Q(v) 1_F$$

A Clifford algebra can also be defined by taking a bilinear form, B, instead of a quadratic form and defining the quadratic form by: $Q(v) = B(v, v)$. One can also construct a bilinear form from a quadratic form via *polarisation*: $B(u, v) = \frac{1}{2}(Q(u+v) - Q(u) - Q(v))$. Using this polarisation we can show that for $u, v \in V$ we have that:

$$uv + vu = 2B(u, v)$$

There is a natural grading of a clifford algebra by the grading $\chi(v_1 v_2 \dots v_k) = (-1)^k v_1 v_2 \dots v_k$ which allows us to decompose a Clifford algebra into even ($\chi = +1$) and odd ($\chi = -1$) parts:

$$\mathrm{Cl}(V, Q) =: Cl^0(V, Q) \oplus Cl^1(V, Q)$$

This decomposition is often useful in technicalities.

B Geometry

A fibre bundle is a very general object and many subclasses are often used in physics such as vector bundle or principal bundles. The bundles of object in spin geometry include algebra bundles along side the usual bundles. A fibre bundle in layman's terms is a way to describe that a space *locally looks like a product space*. If there is are topological spaces E, B, F and a map $f \colon E \to B$, such that for some neighbourhood U of a point $x \in B$, such that $f^{-1}(U)$ is homeomorphic to $U \times F$ in a specific way, then we call (E, B, F, f) a fibre bundle. I include the rigorous definition below for clarity:

Definition B.1. A *fibre bundle* is a structure (E, B, F, f), where E, B, F are all topological spaces and $f \colon E \to B$ is a continuous surjection that satisfies the following properties: $\forall x \in E$ there exists a neighbourhood U of $f(x) \in B$ such that there exists a homeomorphism $\phi : f^{-1}(U) \to U \times F$ which, when composed with projection onto the first component, agrees with f.

The condition for f can be summarised in the requiring that the following diagram commutes:

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & U \times F \arrow[dl, "proj_1"] \\ & U & \end{tikzcd}$$

Given a fibre bundle (E, B, F, f) and given a topological group G that has a left action on the fibres F, then we can specify a G-atlas for the bundle, which is a local trivialisation such that for any overlapping charts (U_i, ϕ_i) and (U_j, ϕ_j) the function:

$$\phi_i \phi_j^{-1} \colon (U_i \cup U_j) \times F \to (U_i \cup U_j) \times F$$

is given by

$$\phi_i \phi_j^{-1}(x, \alpha) = (x, t_{ij}(x)\alpha)$$

where $t_{ij} \colon U_i \cap U_j \to G$ is a continuous map. The maps t_{ij} are called transition functions and given just $(B, F, \{t_{ij}\})$ we can reconstruct the fibre bundle if the transition functions satisfy the following:

$$t_{ii}(x) = x \tag{26}$$
$$t_{ij}(x) = t_{ji}^{-1}(x) \tag{27}$$
$$t_{ik}(x) = t_{ij}(x) t_{jk}(x) \quad \text{on} \quad U_i \cap U_j \cap U_k \tag{28}$$

Definition B.2. A *section* of a fibre bundle (E, B, F, f) is a continuous map $\pi \colon B \to E$ such that $f(\pi(x)) = x$ for all $x \in B$. The space of section of a fibre bundle (E, B, F, f) is denoted by $\Gamma(E)$

Definition B.3. A *fibre bundle morphism* (bundle map) between two fibre bundles (E_1, B_1, F_1, f_1) and (E_2, B_2, F_2, f_2) is defined as a pair continuous maps $\phi\colon E_1 \to E_2$ and $\alpha\colon B_1 \to B_2$ such there exists we have: $f_2 \circ \phi = \alpha \circ f_1$. If we have a bundle map whose inverse is also a bundle map, then we have a bundle isomorphism.

Definition B.4. The *endomorphism* ring over an abelian group A, is denoted by $End(A)$ is the set of all homomorphism of A into itself. If we take A to be a field, and then the *endormorhpism ring* over the space A^n, is an A-algebra and is the set of all linear maps form A^n into A^n. We refer to this endomorphism ring as an *endormorphism algebra*.

C Sanity Checking for the Dirac operator for the Ellipsoid

This is to check that Eq (25) does indeed give the correct Dirac operator we require. We do this by using the equation $D = e^{\alpha a}\sigma_a \nabla_\alpha$. For the ellipse we want to use $\nabla_\alpha = \partial_\alpha + \frac{1}{8}\omega_\alpha^{ab}\left(\gamma_a\gamma_b - \gamma_b\gamma_a\right)$. If we ignore the spin connection terms and concentrate solely on the derivative part we can calculate the $e^{\alpha a}$'s. We do this by looking at the case where the gamma matrices are $\gamma_a = i\sigma_a$, where σ_a are the Pauli matrices. Thus we have the following properties for the gamma matrices:

$$[\gamma_a, \gamma_b] = -2\epsilon_{ab}{}^{c}\gamma_c \qquad \{\gamma_a, \gamma_b\} = -2\delta_{ab} \qquad \gamma_a\gamma_b = -\epsilon_{ab}{}^{c}\gamma_c - \delta_{ab} \tag{29}$$

Using these relations we can simplify Eqn (**??**) to:

$$D_{\mathbb{E}^2} = N\left(\sigma_3\left(\frac{x_1}{\alpha_1^2}\partial_2 - \frac{x_2}{\alpha_2^2}\partial_1\right) - \sigma_2\left(\frac{x_1}{\alpha_1^2}\partial_3 - \frac{x_3}{\alpha_3^2}\partial_1\right) + \sigma_1\left(\frac{x_2}{\alpha_2^2}\partial_3 - \frac{x_3}{\alpha_3^2}\partial_2\right)\right) + \frac{1}{2}\mathrm{div}(n) \tag{30}$$

And thus we can read off the inverse tetrad: $e^{\alpha a}$:

$$e^{\alpha a} = N\begin{pmatrix} 0 & \frac{x_3}{\alpha_3^2} & -\frac{x_2}{\alpha_2^2} \\ -\frac{x_3}{\alpha_3^2} & 0 & \frac{x_1}{\alpha_1^2} \\ \frac{x_2}{\alpha_2^2} & -\frac{x_1}{\alpha_1^2} & 0 \end{pmatrix} \tag{31}$$

Which produces the inverse metric via the formula: $g^{\alpha\beta} = e^{\alpha a}e^{\beta b}\delta_{ab}$

$$g^{\alpha\beta} = N^2\begin{pmatrix} \frac{x_2^2}{\alpha_2^4} + \frac{x_3^2}{\alpha_3^4} & -\frac{x_1}{\alpha_1^2}\frac{x_2}{\alpha_2^2} & -\frac{x_1}{\alpha_1^2}\frac{x_3}{\alpha_3^2} \\ -\frac{x_2}{\alpha_2^2}\frac{x_1}{\alpha_1^2} & \frac{x_1^2}{\alpha_1^4} + \frac{x_3^2}{\alpha_3^4} & \frac{x_2}{\alpha_2^2}\frac{x_3}{\alpha_3^2} \\ \frac{x_3}{\alpha_3^2}\frac{x_1}{\alpha_1^2} & \frac{x_3}{\alpha_3^2}\frac{x_2}{\alpha_2^2} & \frac{x_1^2}{\alpha_1^4} + \frac{x_2^2}{\alpha_2^4} \end{pmatrix} \tag{32}$$

This metric is degenerate and is rank 2. The vector which annihilates this metric is $\left(\frac{x_1}{\alpha_1^2}, \frac{x_2}{\alpha_2^2}, \frac{x_3}{\alpha_3^2}\right)$. Which can be rewritten as: $df = d\left(\frac{1}{2}\left(\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2}\right)\right)$. Using the fact that when this functiion f is constant we can construct a unit normal to surface $f = \text{const}$ by $n = \frac{\nabla f}{|\nabla f|}$. This is what makes us pursue that we have the metric on an ellipsoid in $\mathbb{R}^3$.

So using the parametrisation:

$$x_1 = \alpha_1\cos(u)\sin(v) \tag{33}$$
$$x_2 = \alpha_2\sin(u)\sin(v) \tag{34}$$
$$x_3 = \alpha_3\cos(v) \tag{35}$$

We get the following change of basis matrix (Jacobian)

$$Jac(u,v) = D\left(\frac{u,v}{x_1,x_2,x_3}\right) = \begin{pmatrix} -\frac{\csc(v)\sin(u)}{\alpha_1} & \frac{\cos(u)\csc(v)}{\alpha_2} & 0 \\ 0 & 0 & -\frac{1}{\alpha_3\sqrt{\sin^2(v)}} \end{pmatrix} \tag{36}$$

We note that as v runs between 0 and π then $\sin(v) > 0$ and therefore $\sqrt{\sin(v)^2} = \sin(v)$. So we can write the Jacobian as follows:

$$Jac(u,v) = \begin{pmatrix} -\frac{\csc(v)\sin(u)}{\alpha_1} & \frac{\cos(u)\csc(v)}{\alpha_2} & 0 \\ 0 & 0 & -\frac{\csc(v)}{\alpha_3} \end{pmatrix} \tag{37}$$

We then change the inverse metric $g^{-1}(x_i)$ by the formula:

$$g^{-1}(u,v) = Jac(u,v) \cdot g^{-1}(x_i(u,v)) \cdot Jac(u,v)^T$$

Which gives the following formula:

$$\begin{pmatrix} \frac{\csc^2(v)\left(\alpha_3^2\cos^4(u)+\left(\alpha_1^2\cot^2(v)+2\alpha_3^2\sin^2(u)\right)\cos^2(u)+\alpha_3^2\sin^4(u)+\alpha_2^2\cot^2(v)\sin^2(u)\right)}{\left(\alpha_2^2\cot^2(v)+\alpha_3^2\sin^2(u)\right)\alpha_1^2+\alpha_2^2\alpha_3^2\cos^2(u)} & \frac{(\alpha_1-\alpha_2)(\alpha_1+\alpha_2)\cos(u)\cot(v)\csc^2(v)\sin(u)}{\left(\alpha_2^2\cot^2(v)+\alpha_3^2\sin^2(u)\right)\alpha_1^2+\alpha_2^2\alpha_3^2\cos^2(u)} \\ \frac{(\alpha_1-\alpha_2)(\alpha_1+\alpha_2)\cos(u)\cot(v)\sin(u)}{\alpha_1^2\alpha_2^2\cos^2(v)+\alpha_3^2\left(\alpha_2^2\cos^2(u)+\alpha_1^2\sin^2(u)\right)\sin^2(v)} & \frac{\csc^2(v)\left(\alpha_2^2\cos^2(u)+\alpha_1^2\sin^2(u)\right)}{\left(\alpha_2^2\cot^2(v)+\alpha_3^2\sin^2(u)\right)\alpha_1^2+\alpha_2^2\alpha_3^2\cos^2(u)} \end{pmatrix} \tag{38}$$

This ugly however, if we calculate the inverse of this matrix (as it now has full rank), we should be able to recognise the normal metric for the ellipsoid. And indeed after some trigonometry juggling we arrive at the following metric:

$$g(u,v) = \begin{pmatrix} \left(\alpha_1^2\sin^2(u)\alpha_2^2\cos^2(u)\right)\sin^2(v) & (\alpha_2^2-\alpha_1^2)\cos(u)\cos(v)\sin(u)\sin(v) \\ (\alpha_2^2-\alpha_1^2)\cos(u)\cos(v)\sin(u)\sin(v) & \alpha_3\sin^2(v)+\cos^2(v)\left(\alpha_1^2\cos^2(u)+\alpha_2^2\sin^2(u)\right) \end{pmatrix} \tag{39}$$

Huzza!

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