Non-commutative Geometry and Gravity Models

How certain gravity models arise from the non-commutative geometry (NCG) of particle physics and how to approach Quantum Gravity within this framework.

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Introduction to Noncommutative Geometry

Noncommutative Geometry à la Connes

Geometric		**Algebraic**
Manifold M	$\Longleftrightarrow$	Algebra $A = C^\infty(M)$
Riemannian metric g		Hilbert Space $\mathcal{H} = L^2(S)$
Spinor bundle $S \to M$		Dirac Operator $\not{D} = i\gamma^a {e_a}^\mu \nabla^S_\mu$

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Extend to Noncommutative Algebras

Make A noncommutative but keep all other requirements the same

Are their noncommutative manifolds?

What are their properties?

Particle Physics From Noncommutative Geometry

Particle Physics as a NCG

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$$\left(\overbrace{\mathbb{C}^4}^{\nu_R,\nu_L,e_R,e_L} \oplus \left(\underbrace{\mathbb{C}^4}_{u_R,u_L,d_R,d_L} \otimes \underbrace{\mathbb{C}^3}_{\text{colour}} \right) \oplus \overbrace{\mathbb{C}^4}^{\bar{\nu}_R,\bar{\nu}_L,\bar{e}_R,\bar{e}_L} \oplus \left(\underbrace{\mathbb{C}^4}_{\bar{u}_R,\bar{u}_L,\bar{d}_R,\bar{d}_L} \oplus \underbrace{\mathbb{C}^3}_{\text{color}} \right) \right)^{\overbrace{\oplus 3}^{\text{generations}}}$$

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$D_F =$ matrix of Yukawa couplings and Majorana masses

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- What is the spectral action?
- What are the other gravitational terms?

Spectral Action

Connes-Chamseddine Spectral Action

- $S(D,\Omega) = Tr\left(f[D/\Omega]\right)$
- f is a smooth approximation of a cut-off function to regularise the trace
- It's moments ($f_i = \int_0^\infty f(v)v^{i-1}dv$) will play a role in determining which gravity theory is selected

Fermionic Action

- $S(D,\zeta) = \frac{1}{2}\left(J\zeta, D\zeta\right)$
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Combined the two: $S[D,\Omega,\zeta] = Tr(f[D/\Omega]) + \frac{1}{2}(J\zeta, D\zeta)$

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$$+ \frac{f_0}{10\pi^2}\int\Big(\frac{11}{6}\underbrace{R^*R^*}_{\text{Gauss-Bonnet}} - \underbrace{3C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}_{\text{Conformal gravity}}\Big)\sqrt{g}\,d^4x$$

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Rewrite them in terms of effective constants depending on the parameter Ω.

Modified Gravity Theories from Spectral Action

$$Tr(f[D/\Omega])^{\Omega\to\infty}\!\!\gtrsim \int d^4x \Bigg(\overbrace{\frac{1}{16\pi G_{eff}(\Omega)} R\sqrt{g}}^{\text{Einstein-Hilbert}} + \overbrace{\frac{\Lambda_{eff}(\Omega)}{8\pi G_{eff}(\Omega)}\sqrt{g}}^{\text{Cosmological Constant}}$$

$$+ \tau_0(\Omega) \underbrace{R^* R^*}_{Gauss-Bonnet} \sqrt{g} + \alpha_0(\Omega) \underbrace{C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}}_{\text{Conformal Gravity}} \sqrt{g} \Bigg)$$

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Need to go deeper

Noncommutative Geometry Approaches to Quantum Gravity and Unification

Quantum Gravity Theories

How to make spacetime noncommutative?

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  2. For noncompact Lie group symmetry (like Lorentz symmetry) it's unclear and needs more work

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Exactly how is to be seen as we need to consider the following:

Fuzzy Spaces

My work is similar to that of James Gaunt's.

Main Projects

- Fuzzy versions of coadjoint orbits (including Dirac operators)
- Geometrical Measurable quantities on fuzzy spaces (dimension ✓, volumes ✓, curvature?)
- The path integral over Dirac operators. What is a good action? What are the features?
  - Opportunity for machine learning to be applied

Outlook and further work

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Wants

- Notion of convergence of noncommutative spaces under a commmutative limit (James Gaunt's talk from yesterday)
- Extending higher structures to a noncommutative setting and enforcing compatibility with commutative analogs (recall need noncommutative gauge equivalence)
- Method for defining Noncommutative nonsymmetric spaces

Thank you for your attention

Any questions?