Fuzzy geometry and spectral zeta functions

A quest for the meaning of dimension in noncommutative geometry.

By Paul Druce
University of Nottingham, UK

Outline

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- Introduction to fuzzy geometry

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- Zeta functions and Heat Kernel Expansions

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- What are some spectral observables? (and do they work for fuzzy geometries?)

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- What are some spectral observables? (and do they work for fuzzy geometries?)
- Investigations into fuzzy spectral zetas as a dimension measure.

Commutative Geometry in NCG Framework

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$(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ is a *real spectral triple* with $\mathcal{A}$ unital and commutative

Then there exists [1] a Riemannian manifold (M, g) with spin structure (S, J_M) such that $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ is given by

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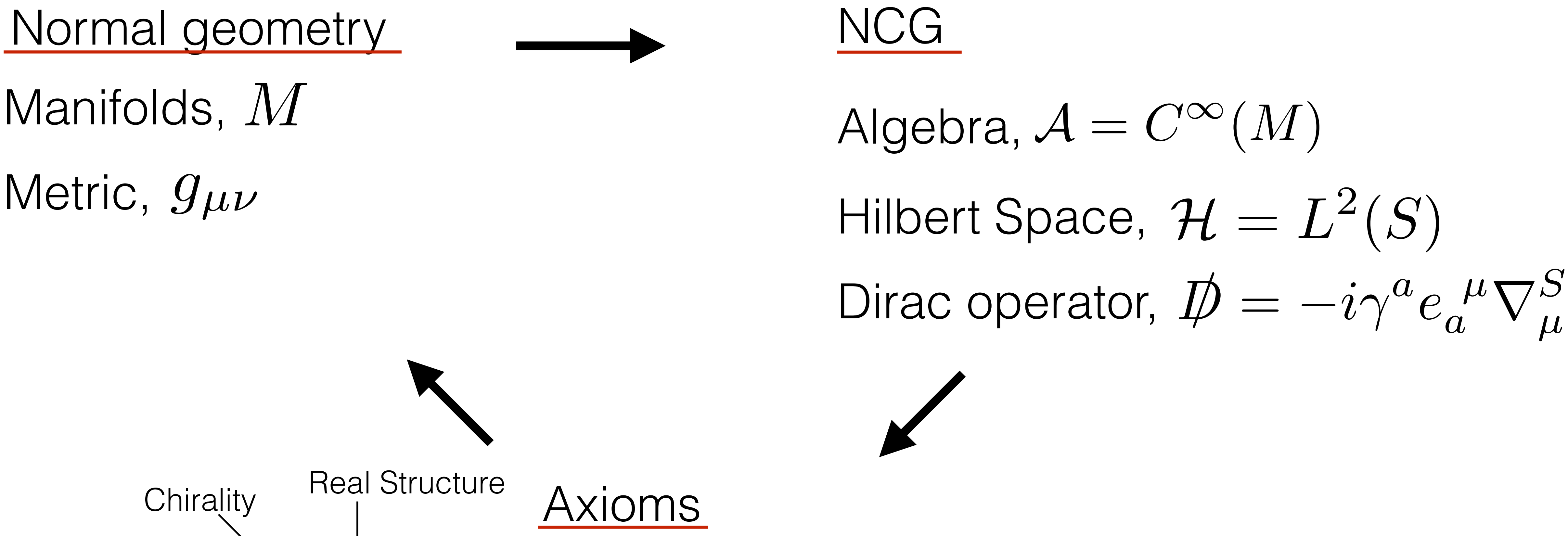

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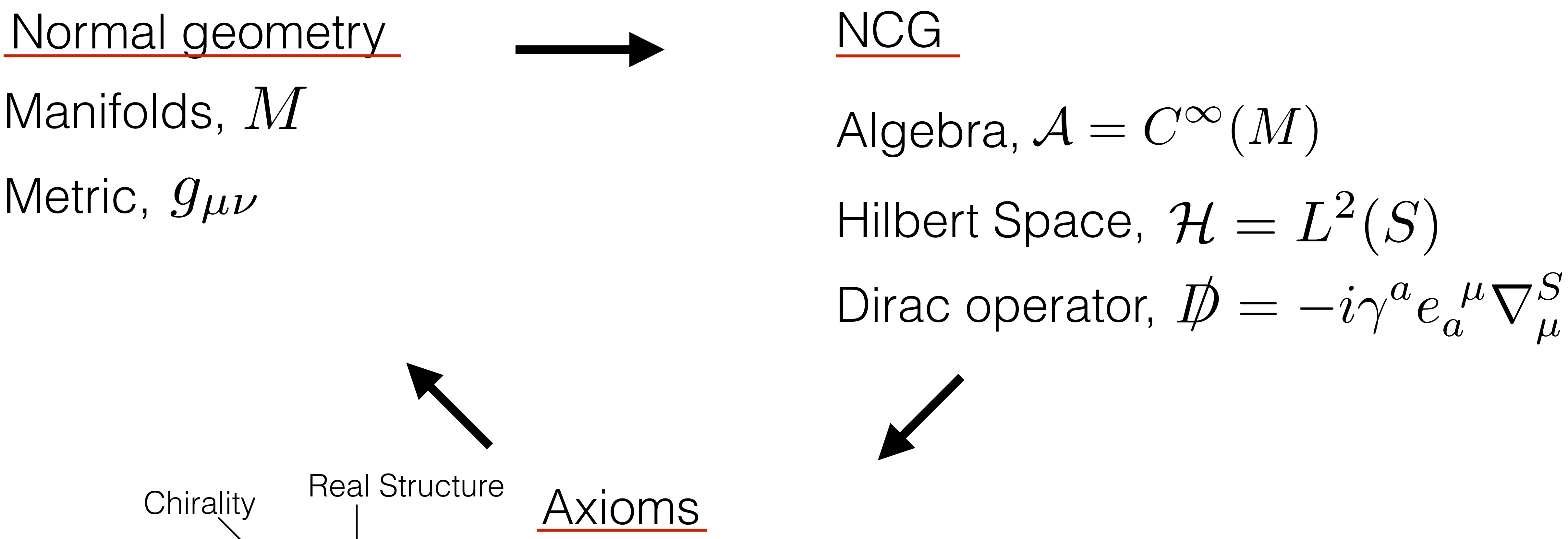

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- Make $\mathcal{A}$ a noncommutative algebra
- Can we find noncommutative ‘manifolds’
- If so what are some of their properties?

Finite Spectral Triples

[1] Barrett, J. W. (2015). Matrix geometries and fuzzy spaces as finite spectral triples. Journal of Mathematical Physics, 56(8), 082301. http://doi.org/10.1063/1.4927224

Finite Spectral Triples

- Take $\mathcal{A}$ to be: $M_n(\mathbb{C})$
- Take $\mathcal{H}$ to be: $V \otimes M_n(\mathbb{C})$

$$\{\gamma^i, \gamma^j\} = 2\eta^{ij}$$

$$\eta = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q})$$

$$V = \mathbb{C}^{p+q}$$

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$$D(v \otimes m) = \sum_i \alpha^i v \otimes [L_i, m] + \sum_j \tau^j v \otimes \{H_j, m\}$$

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Random Fuzzy Geometries: J Barrett & L Glaser [1]

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Fuzzy Torus: J Barrett & J Gaunt [2]

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Spectrum of Operators

(specifically the Dirac operator)

Spectral Zeta Function

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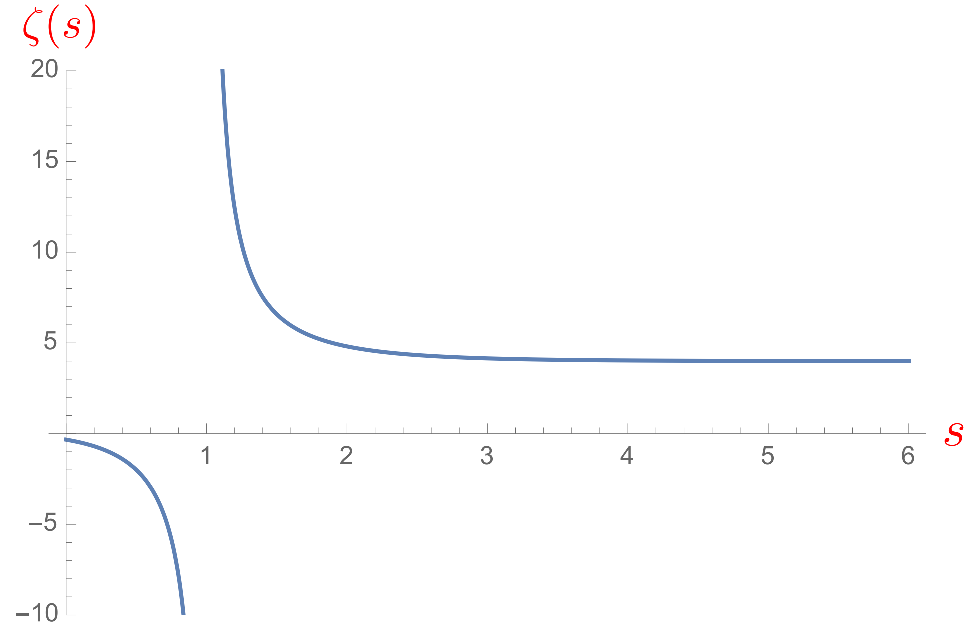
Sphere:

$$spec(\not{D}_{\mathbb{S}^2}) = \mathbb{Z}$$

$$\zeta_{\not{D}^2}(s) = \sum_{n=1}^{\infty} 4n \frac{1}{n^{2s}} = 4 \cdot \zeta_R(2s-1)$$



Back to fuzzy geometry

Fuzzy Problem

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Look for the beginnings of a divergence?

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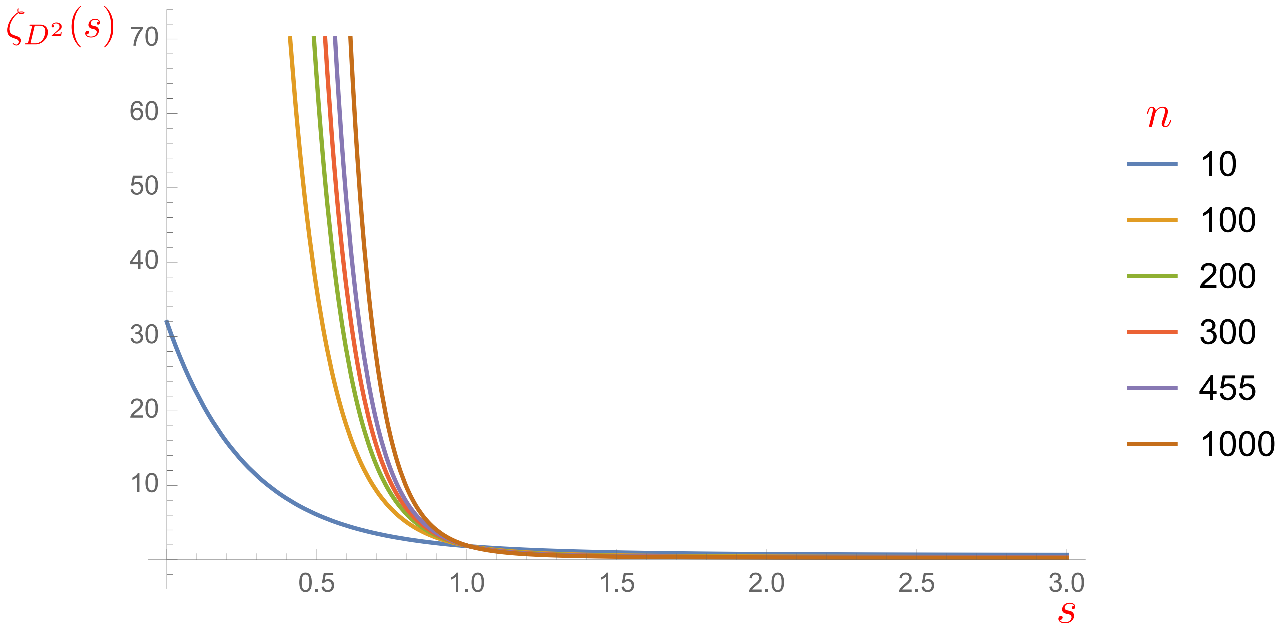

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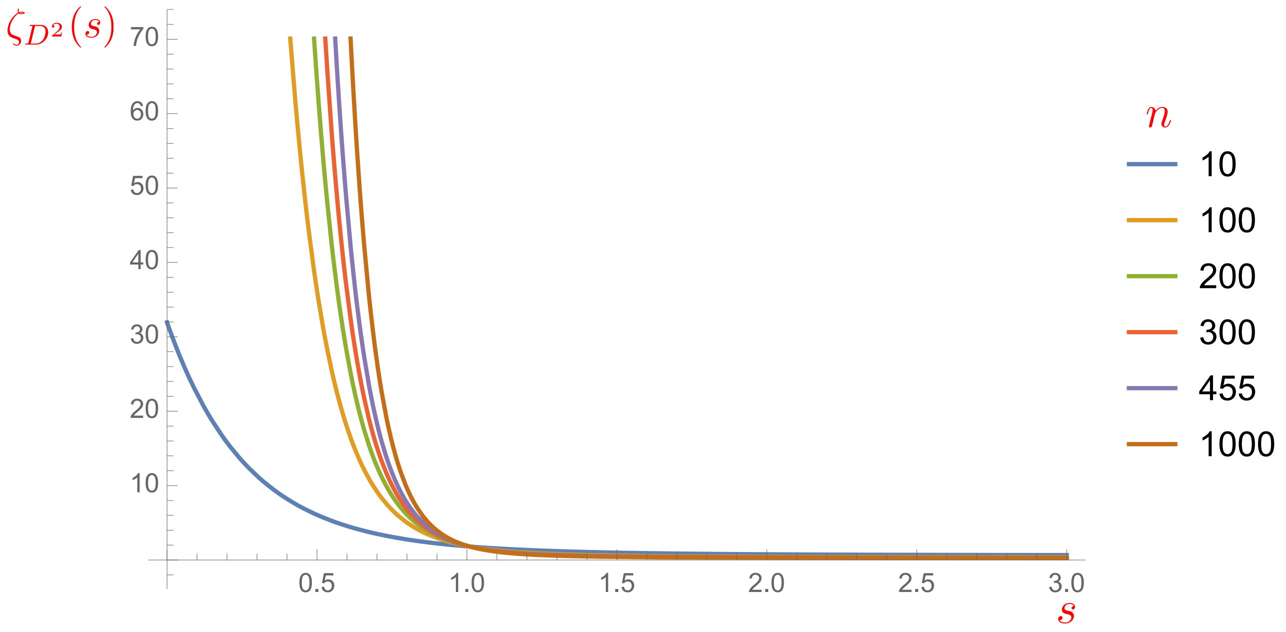

Hard to see.

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Fuzzy Zeta Function?

$$\zeta_{D^2}(s) = \sum_i \left(\lambda_i^2\right)^{-s} \xrightarrow{s=0} \zeta_{D^2}(0) \sim \sum_i 1 = \#\{\lambda_i\}$$

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$$\frac{\zeta_{D^2_n}(s)}{\log(\zeta_{D^2_n}(0))} = \frac{\zeta_{D^2_m}(s)}{\log(\zeta_{D^2_m}(0))}$$

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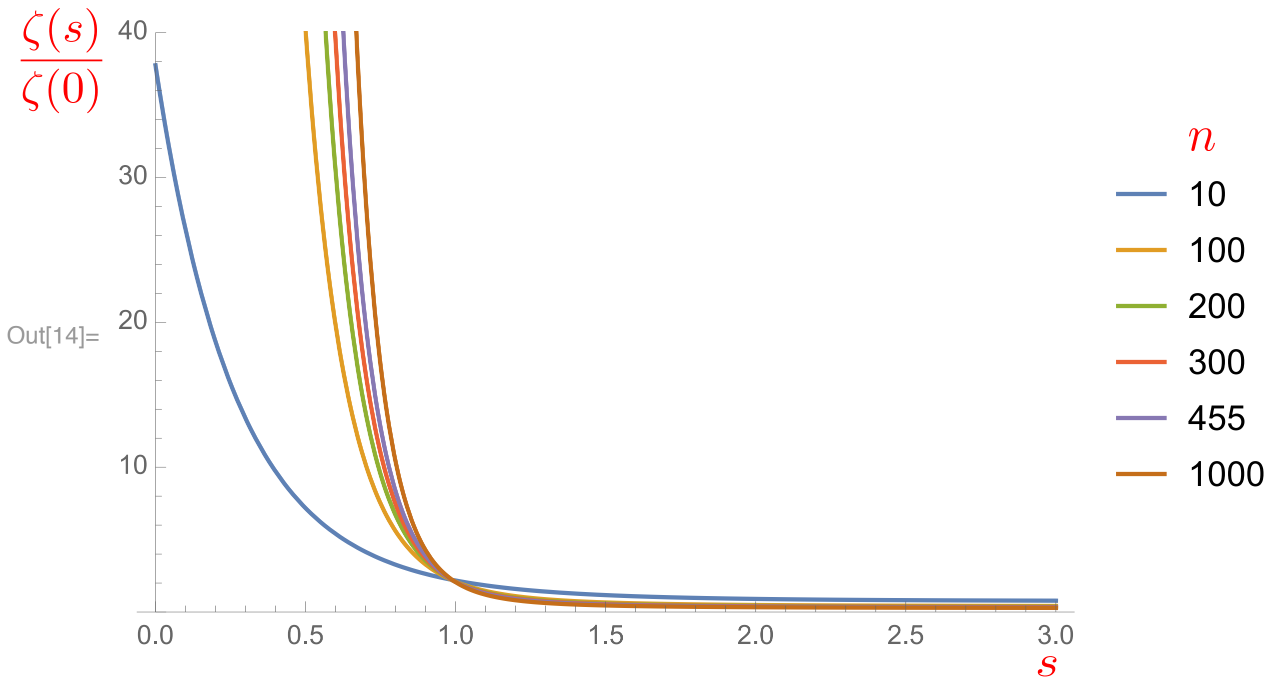

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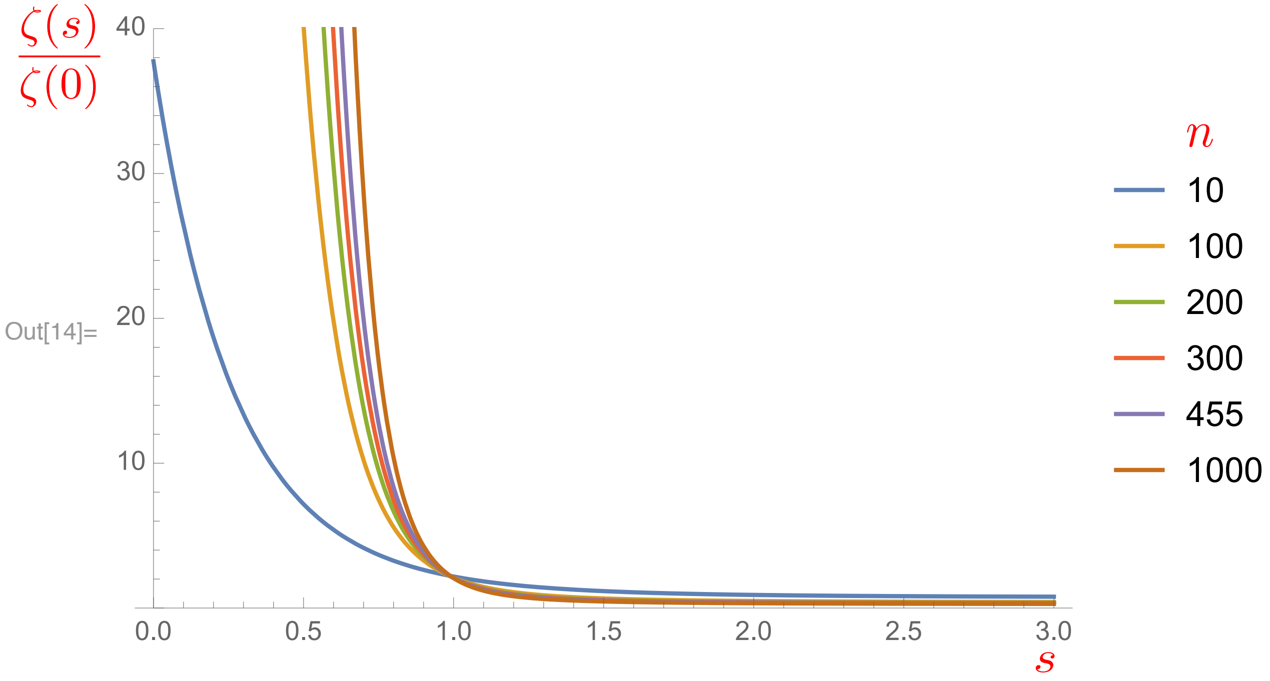

$$f(n,m) = \frac{\zeta_{D_n^2}(s)}{\log(\zeta_{D_n^2}(0))} - \frac{\zeta_{D_m^2}(s)}{\log(\zeta_{D_m^2}(0))}$$

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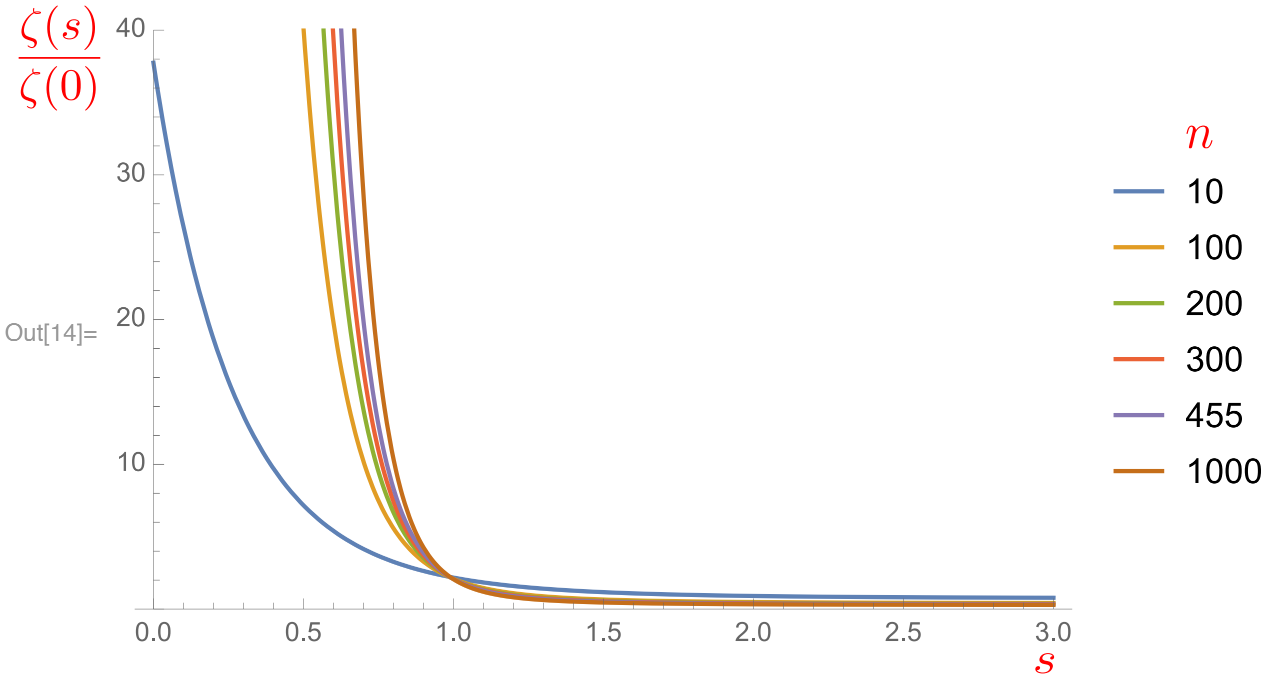

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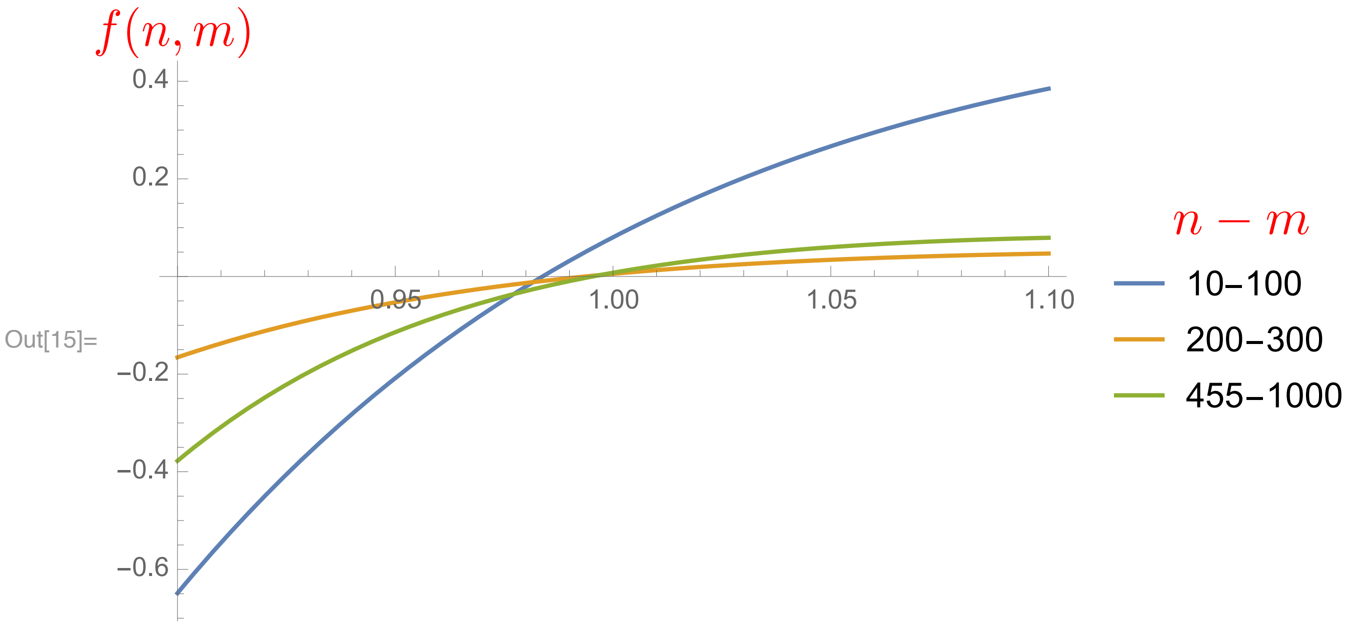

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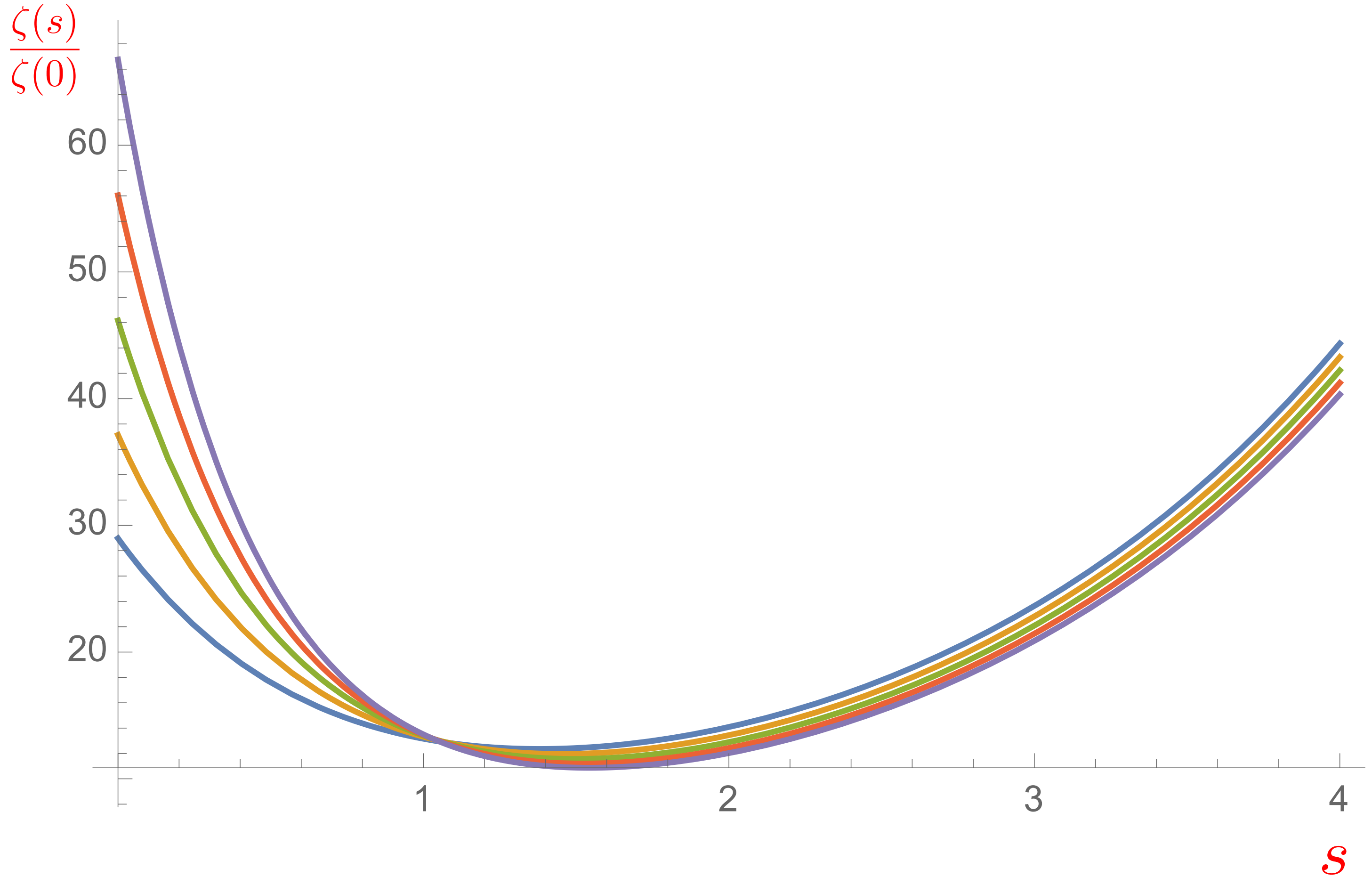
Fuzzy Torus:

paper to appear by J Barrett and J Gaunt



Fuzzy Zeta Function?

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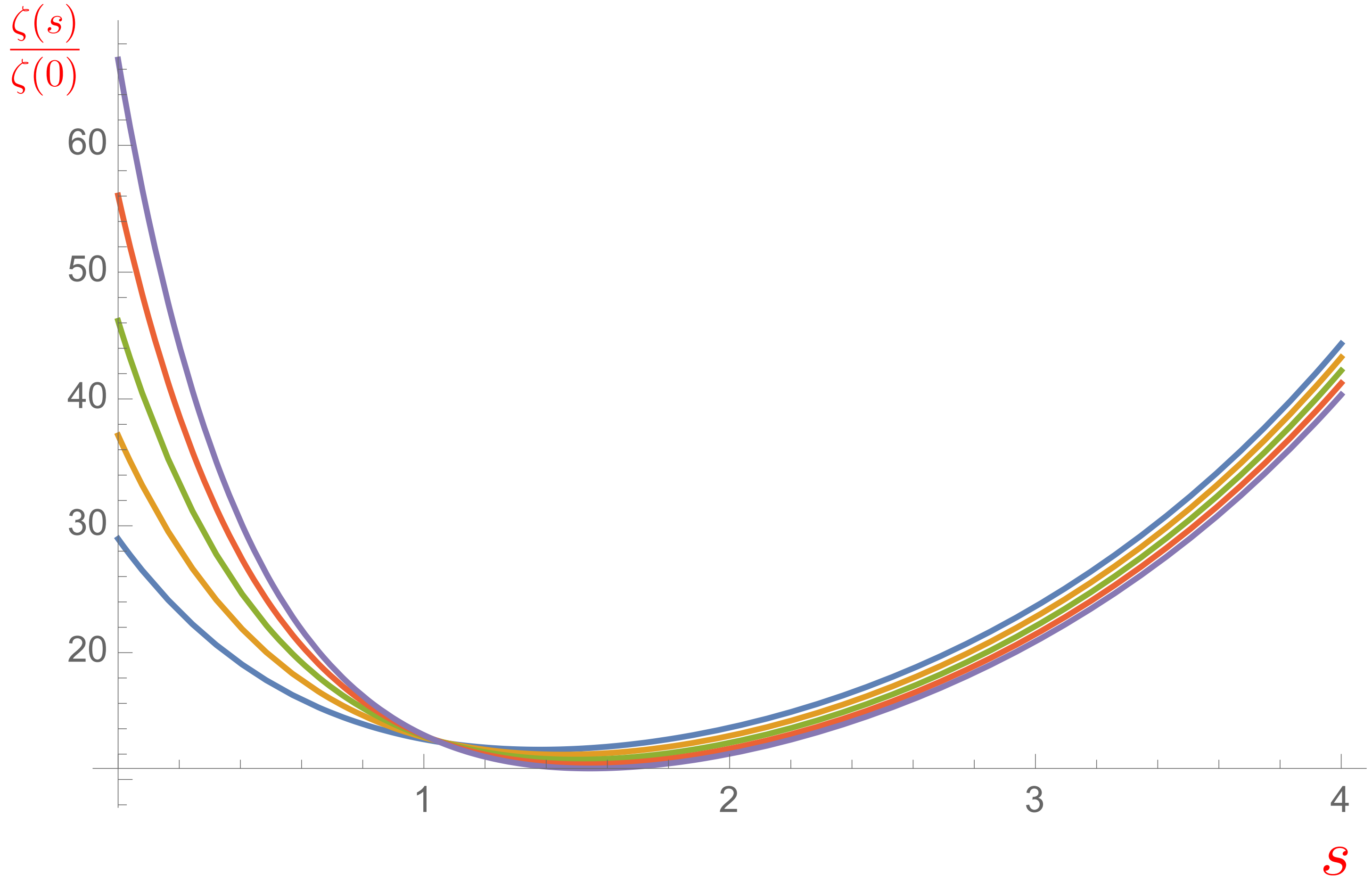

Need more examples, but good so far

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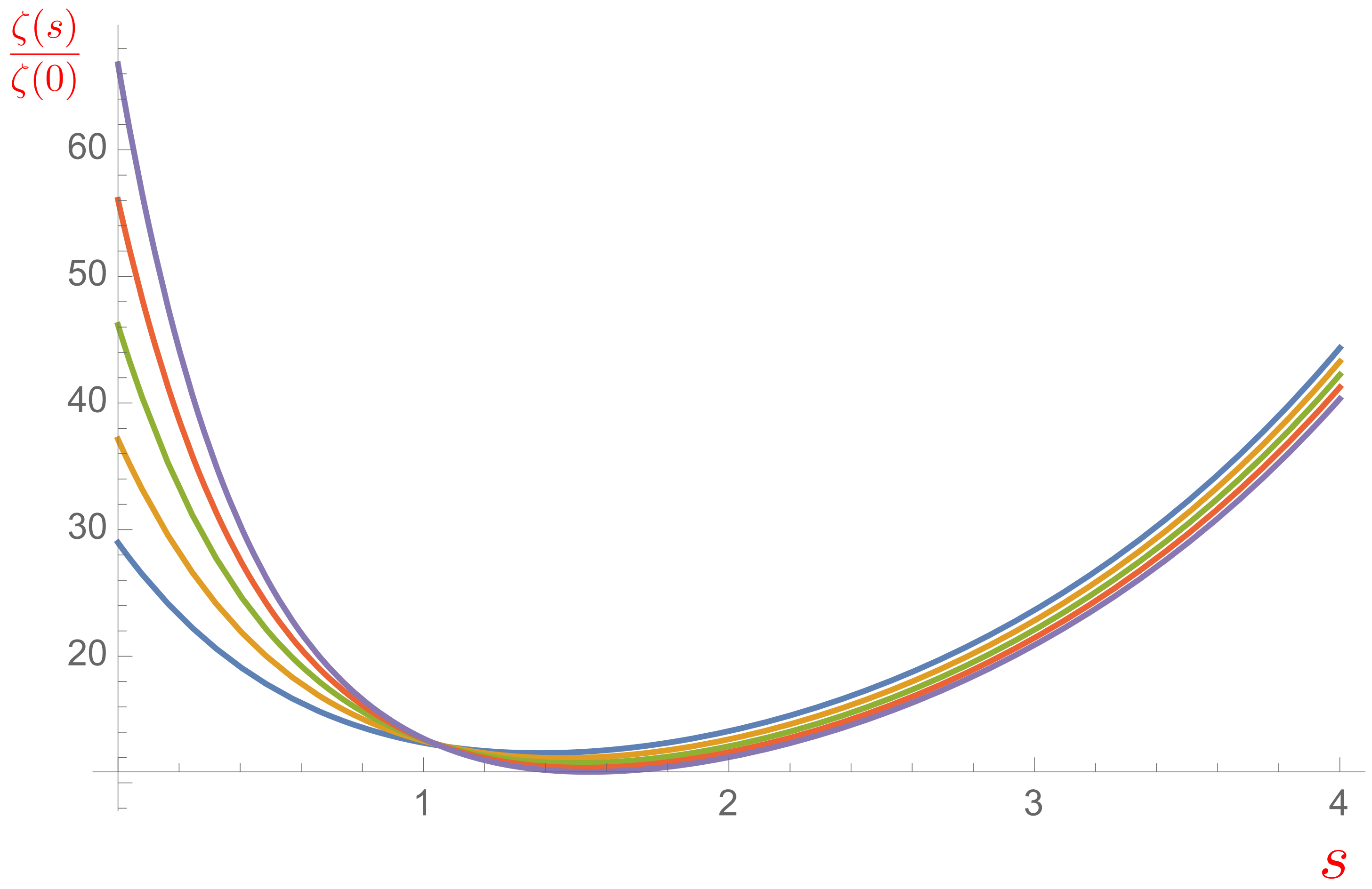

Need more examples, but good so far

What else can we do?

Back to normal geometry

Heat Kernel Expansion

Vassilevich, D. V. (2003). Heat kernel expansion: user's manual. Physics Reports, 388(5), 279–360. http://doi.org/10.1016/j.physrep.2003.09.002

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Heat Kernel Expansion

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Heat Kernel Expansion

What?

General Laplace type

$$K(t;L) = \mathrm{Tr}(e^{-tL})$$

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$\xrightarrow{\text{Invert}}$

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$$\boxed{a_k(D^2) = Res_{s=\frac{n-k}{2}} (\Gamma(s)\zeta_{D^2}(s))}$$

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Who cares?

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Back to normal geometry

Geometry from the expansion

$$a_0(D^2) = \frac{1}{(4\pi)^{\frac{n}{2}}}\mathrm{Tr}(Id)\int_M d^n x\sqrt{g}$$

$$a_2(D^2) = \frac{1}{(4\pi)^{\frac{n}{2}}}(-\frac{1}{12})\mathrm{Tr}(Id)\int_M d^n x\sqrt{g}\underset{\substack{\uparrow \\ \text{Ricci scalar}}}{R}$$

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Over spinor space

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$$a_0 = \frac{Vol(M)}{2\pi}$$

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$$a_k(D^2) = Res_{s=\frac{n-k}{2}}(\Gamma(s)\zeta_{D^2}(s))$$

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Geometry from the expansion

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$$a_0(D^2) = \frac{1}{(4\pi)^{\frac{n}{2}}}\mathrm{Tr}(Id)\int_M d^n x\sqrt{g} \qquad a_2(D^2) = \frac{1}{(4\pi)^{\frac{n}{2}}}(-\frac{1}{12})\mathrm{Tr}(Id)\int_M d^n x\sqrt{g}R$$

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$$Vol(M) = 2\pi Res_{s=1}(\Gamma(s)\zeta_{D^2}(s))$$

$$\chi(M) = -6Res_{s=0}(\Gamma(s)\zeta_{D^2}(s)) = -6\zeta_{D^2}(0)$$

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Fuzzy Residue?

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Fuzzy Residue?

$$Vol(M) = 2\pi Res_{s=1}(\Gamma(s)\zeta_{D^2}(s))$$

How to calculate this when there is no pole?

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Guess: Read off value of: $\dfrac{\zeta_{D_n^2}(1)}{\log(\zeta_{D_n^2}(0))}$

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Sphere:

n	$\frac{\zeta_{D^2_n}(1)}{\log(\zeta_{D^2_n}(0))}$
10	2.1735
100	2.09315
200	2.08172
300	2.07624
455	2.07133
1000	2.06359
∞	2

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Zeta function continuation

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Everything is analytic to begin with.

So what is analytic continuation in the fuzzy world?

Summary

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- Used spectral zeta to find dimension of fuzzy geometries

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- Used spectral zeta to find dimension of fuzzy geometries
- Used spectral zeta to find ‘volume’ of fuzzy spheres

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- Construct more fuzzy spaces to test on

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Todo

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- Figure out ‘analytic continuation’
- What about distinguishing isospectral spaces?

Thank you for listening!

Any questions?